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Minimization of curvature in Conformal Geometry

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Minimization of Curvature in Conformal Geometry

submitted by

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for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

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Abstract

Let (M, g_0) a smooth compact Riemannian manifold with smooth boundary and dimension $n \geq 3$. We consider a minimization problem for the scalar curvature R after a conformal change of the form $g = u^{2^*-2}g_0$, where u is a smooth positive function on M . In particular, we seek for minimizers of the $\|\cdot\|_\infty$ functional of R , within a conformal class, under small energy assumptions and natural geometric constraints, in order to generalize a result of Moser and Schwetlick in the case of surfaces with boundary.

The nonreflexiveness of the space $L^\infty(M)$ forces us to first study the corresponding minimization problem for the L^p norm of R after a conformal change. We establish the existence of a priori bounds for solutions of our equation under our assumptions. Then we prove the existence of minimizers for the associated p -problem via the Direct Method. The 4th-order Euler Lagrange equations are derived and studied, as well as regularity properties of their solutions, for all $p \in (q, \infty)$, where $q > \frac{n}{2}$.

Finally we let $p \rightarrow +\infty$ and show that the limit equation produces a minimizer for our original problem. Moreover we study the structure of the nodal set Γ of a solution of the limit Euler Lagrange equation. We draw a connection between the form of the curvature of the minimizer and this nodal set. We specifically show, that the minimizer will have constant scalar curvature, outside of the set Γ , thus obtaining a connection with the Yamabe Problem on manifolds with boundary.

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CHAPTER 1

INTRODUCTION

A central concept in the field of Riemannian Geometry is that of curvature. In particular, given a smooth compact Riemannian manifold with boundary (M, g_0) , there are various curvature tensors associated to it, used to measure the extent to which the manifold M is not flat. Namely, we can try to calculate the Riemann, Ricci, sectional or scalar curvature of M with respect to the Riemannian metric g_0 . Among them we can think of the scalar curvature R_0 of g_0 to be a simpler notion, due to the fact that it is a scalar function. In particular, in local coordinates ([5]), using the Einstein summation convention, we can write:

$$R_0 = g^{ij} R_{ij}.$$

Here, g^{ij} are the components of the inverse of the metric tensor in local coordinates and R_{ij} these of the Ricci tensor.

The scalar curvature of a manifold locally measures the extent to which the manifold deviates from being flat, by comparing the volumes of certain objects. If a geodesic ball encloses more volume than a Euclidean ball of the same radius, then locally the scalar curvature will be negative, and vice versa. Hence, if at a point $p \in M$ it holds that $R_0(p) < 0$, we have:

$$Vol(B_R(p), g_{eucl}) < Vol(B_R(p), g_0),$$

for R small enough, where g_{eucl} stands for the standard flat metric on \mathbb{R}^n . This intuition is gained by looking at some asymptotic expansions for the volume of a small ball around p , see [10] for example.

A relatively simpler concept is that of a conformal change of metric. This is a change of metric that preserves the angles between two tangent vectors, but distorts the lengths of them. In particular, bearing in mind that the angle θ between two tangent vectors $X, Y \in T_p M$ is given by:

$$\theta = \arccos\left(\frac{g(X, Y)}{\sqrt{g(X, X)g(Y, Y)}}\right),$$

any change of the form $\tilde{g} = \lambda g$ for $\lambda > 0$ does not change this angle.

The aforementioned conformal change of metric, also known as pointwise conformal change, is a special case of a more general equivalence relation between conformal metrics known as conformal equivalence. More precisely, we give the following definition (see [22]). The smooth metrics g_1, g_2 are conformally equivalent, if there exists a diffeomorphism $p : M \rightarrow M$ such that:

$$p^* g_1 = m g_2,$$

for a smooth positive function m on M .

Particular choices of conformal factors lead to nice transformation laws for geometric quantities before and after a conformal change. If we let $u : \bar{M} \rightarrow \mathbb{R}$ be a smooth positive function, we can consider conformal changes of the form

$$g = u^{2^*-2} g_0, \tag{1.0.1}$$

where $2^* = \frac{2n}{n-2}$ is the critical exponent for the Sobolev Embedding Theorem [31], if $n > 2$. We also note that, in the case of surfaces, one normally uses a conformal change of the form

$$g = e^{2f} g_0, \tag{1.0.2}$$

for a smooth real valued function f on our surface. For a list of transformation laws after a conformal change, we refer to [7].

A starting point for the problem we will consider in this work, is the transforma-

tion of scalar curvature after a conformal change. In particular, after a conformal change of metric of the form (1.0.1) the scalar curvature R_0 transforms under the law (see [23]):

$$-c_n \Delta_{g_0} u + R_0 u = R u^{2^*-1}, \quad (1.0.3)$$

where Δ_{g_0} is the Laplace-Beltrami operator of g_0 , $c_n = \frac{4(n-1)}{(n-2)}$ and R is the scalar curvature of g . A remark that can be made immediately, is the presence of the critical exponent nonlinearity on the righthandside, which is of significant importance, as we will see later on. Furthermore, we note that on a Riemann surface, the corresponding equation for a change of the form (1.0.2) is (see e.g. [22]) :

$$-\Delta_{g_0} f + K_0 = e^{2f} K,$$

where K_0 and K stand for the Gaussian curvature of g and g_0 respectively. Note that in the surface case we get a different type of nonlinearity, in particular one of exponential type, suggesting that a different approach is needed.

The behavior of various geometric quantities on a Riemannian manifold, and especially that of its scalar curvature, under a conformal change of metric have been studied in great detail in the past as parts of various problems. A common question asked in that context, is the following: Does there exist a best metric conformal to g_0 ?

Here, "best" is understood in the sense of having some particularly nice property. An attempt in that direction is the so called Yamabe Problem [23], now known as the Yamabe Theorem: in the 1960s Yamabe [34] tried to prove an analogue of the Uniformization Theorem for manifolds without boundary, with dimension strictly greater than 2. In his work, he claimed to have a proof of the fact that under a conformal change, we can find a Riemannian metric of constant scalar curvature. From a PDE point of view, this amounts to the existence of a smooth positive solution to the equation:

$$-c_n \Delta_{g_0} u + R_0 u = c u^{2^*-1},$$

with $c \in \mathbb{R}$.

The Uniformization Theorem leads to a classification of Riemann Surfaces up

to conformal equivalence (see [1] for example), so the result Yamabe tried to prove, would lead to similar results in the higher dimensional case. More precisely, the Uniformization Theorem states that for every compact surface, we can always find a conformal metric of constant Gauss curvature. So, a proof from Yamabe could lead to a similar classification of manifolds in the higher dimensional case.

Neil Trudinger [33] found a gap in Yamabe's proof and proved the result Yamabe had stated, under a smallness assumption on the total scalar curvature of the manifolds under study. Aubin [3] then provided a proof in the case that the manifold has dimension $n \geq 6$ and is not locally conformally flat. Finally, Schoen [28] completed the proof in the remaining cases. A unified approach for the Yamabe problem can be found in [23].

In the case of manifolds with boundary Escobar [13], [14] proved similar results, studying the Yamabe problem under boundary conditions involving the mean curvature of the boundary of the manifold. If we call R_g and h_g the scalar and mean curvatures of a conformal metric g respectively, we can ask whether:

1. R_g is constant and h_g zero
2. R_g zero and h_g constant.

This amounts to proving existence results for a partial differential equation as in the closed manifold case, but with a nonlinear boundary condition. Namely, if we denote by h_0 and h_g the mean curvature of ∂M before and after a conformal change respectively, we need to establish existence for an equation of the form (1.0.3), under boundary conditions like:

$$\frac{n-2}{2}h_g u^{\frac{n}{n-2}} = \frac{n-2}{2}h_0 u + \frac{\partial u}{\partial \nu_0}.$$

Moreover, Marques [24], [25] considered some cases not answered by Escobar, thus giving more general results. In addition, there is the work of Han and Li [17], where using different methods, they proved the existence of conformal metrics with constant scalar and mean curvature.

A related problem is that of prescribed scalar curvature, known as the Nirenberg problem for surfaces. In that context, it is asked if a certain smooth function

can be the scalar curvature function of a Riemannian manifold, after a conformal change (see [22], [12] for example). Both the Yamabe and prescribed scalar curvature problems are related to the problem we will study.

A different approach to the aforementioned results was proposed in [26]. In particular, a minimization problem for a weighted variant of the Gaussian curvature after a conformal change was studied, on a compact smooth surface (S, g_0) with smooth boundary ∂S . If g is a metric conformal to g_0 , with $g = e^{2f}g_0$, using formula (1.0.3), we have the Gaussian curvature of g defined via:

$$K = e^{-2f}(-\Delta_0 f + K_0).$$

After the introduction of a smooth positive weight function $k : S \rightarrow \mathbb{R}$, the authors prove that the functional:

$$E_\infty(f) = \operatorname{ess\,sup}_S \frac{|K|}{k}$$

attains its infimum on a suitably defined set. This is valid provided that certain bounds hold on the energy of the minimizer, in the class of functions they study. Moreover, some additional information on the Gaussian curvature K_m of the minimizer f_m is obtained. Namely, it is proved that

$$|K_m| = E_\infty(f_m)|k|,$$

outside of a set Γ . A precise characterization of Γ is given, as the nodal set to a solution of a linear second order elliptic equation. This provides a certain connection to the prescribed Gaussian curvature problem.

1.1 Outline

The aim of this work is to extend the results of [26] in the higher dimensional case. In more detail, we consider a smooth compact manifold M with smooth boundary ∂M , equipped with a smooth metric g_0 . As we already noted, a conformal change of the form:

$$g = u^{2^*-2}g_0,$$

gives rise to the scalar curvature R_g satisfying equation (1.0.3).

We prove that in a fixed conformal class $[g_0]$, we can find a metric g minimizing the functional

$$E(u) = \operatorname{ess\,sup}_M |R_g|,$$

under some suitably chosen constraints and boundary conditions described later. In particular, we show that if we impose a sufficiently small upper bound on the infimum of E , defined on a suitably restricted set of functions satisfying certain constraints, we can find a critical metric. Moreover, the scalar curvature of the minimizing metric will be locally constant, outside of a small set with a precise description, similarly to the lower dimensional case in [26]. Thus, a connection to the problems studied by Escobar [13], [14] is made, as we might end up with a constant scalar curvature metric on a manifold with boundary.

The organization of our work is as follows. We begin in Chapter 2, by introducing all the necessary notions and results from Riemannian Geometry and Partial Differential Equations needed for this work. Moreover, we provide all the transformation laws after a conformal change for the geometric objects under consideration. Finally, we give an overview of the Direct Method in our context, and its connections with the Yamabe problem on closed manifolds, as well as on manifolds with boundary.

In Chapter 3, we state our main result. We introduce the method that we will use, namely approximation of the $\|\cdot\|_\infty$ norm by the limit as $p \rightarrow \infty$ of $\|\cdot\|_p$ norms. Then, in Chapter 4, we provide proofs for the existence of upper and lower bounds of solutions to our equation. These are obtained via a blow-up type analysis of solutions to our equation, and some applications of the maximum principle respectively. Moreover, we briefly discuss connections of our work to existing results related to the blow-up theory of elliptic equations in a geometric context.

In Chapter 5, we show that there exists a $q > 0$ such that for $p \geq q$ we can establish existence of a minimizer for the p -problem. We derive the first variation of the functionals that we study, and calculate the Euler Lagrange equations explicitly. Some regularity properties for the solutions of the Euler Lagrange equations are derived, allowing us to establish good convergence modes.

After using those, we can then pass to the limit, and study a limit equation in Chapter 6. Finally, some regularity results on the nodal domains of solutions to some semilinear equations related to our problem are studied, leading to similar results to [26] in the lower dimensional case, thus concluding our work.

CHAPTER 2

BACKGROUND MATERIAL

In this Chapter we will introduce all the necessary notation and notions for the rest of our work. References to the relevant literature will be given in each individual section.

2.1 Notation

We begin our Chapter of background material, by introducing some notation conventions that will be followed subsequently in this text, without further notice. In what follows, for all quantities related to the background metric g_0 , we will use 0 or g_0 as a subscript, in order to differentiate them from quantities related to its conformal metric g . The Einstein summation convention is also used, when two repeated indices occur. Also, the letter C with various subscripts will denote a constant, unless otherwise stated. A tilde over a symbol will be used, in order to distinguish notions defined on M extended to the boundary ∂M , if necessary. We will denote the gradient with respect to g_0 by ∇ , and the Levi-Civita connection by ∇^0 making no further remarks, unless when further clarification is needed. The symbol ∇^k will denote the k – *th* order covariant derivative of a function. Instead of writing $\tilde{g}_0(N_0, \nabla f)$ for the normal derivative of a function f in the direction N_0 , we will use $\frac{\partial f}{\partial \nu_0}$.

2.2 Manifolds with Boundary and their Curvature Tensors

For the sake of completeness, we will first give the necessary definitions from Riemannian Geometry, as well as some transformation laws for some of them after a conformal change of metric. We begin by defining manifolds with boundary, which are the central geometric objects of our study. Most the results presented here, can be found in most Riemannian Geometry or Geometric Analysis texts, see [5], [6] or [21] for example.

Definition 2.2.1. *A second countable, Hausdorff, paracompact topological space M is a manifold with boundary, if there exists a family $\mathcal{U} = (U_i, \phi_i)$ of open sets $\{U_i\}_{i \in I}$ that cover of M and homeomorphisms $\phi_i : U_i \rightarrow V_i$, mapping to open sets $V_i \subset \overline{\mathbb{R}_+^n} = \overline{\{x \in \mathbb{R}^n, x_1 > 0\}}$.*

Following standard notation we call \mathbb{R}_+^n the half space. In order to have a visual representation of a manifold with boundary, we can identify the interior of M as the subset of M consisting of points having neighborhoods homeomorphic to open sets of \mathbb{R}^n , and the boundary of M as the remaining part of M . In particular, its boundary is itself a manifold without boundary. As typical examples of manifolds with boundary we can consider closed Euclidean balls:

$$\overline{B}_r(0) = \{x \in \mathbb{R}^n, |x| \leq r\}.$$

Without loss of generality, we can assume that $r = 1$. Then, the open ball of radius 1 is homeomorphic to itself, and is an open set of \mathbb{R}^n . The boundary of $\overline{B}_1(0)$ is the sphere S^{n-1} of center 0 and radius 1. Since S^{n-1} is a smooth $(n - 1)$ dimensional manifold itself, we can always find homeomorphisms ϕ_i defined on neighborhoods of its points mapping to open sets in \mathbb{R}^{n-1} .

In the rest of the text, we will work on a smooth compact manifold M , that is, a manifold with smooth transition maps. Moreover, we will assume that M has a smooth boundary. For each point p , we will denote the tangent space to p by $T_p M$ (for a definition see [21]). The tangent bundle to M is the smooth manifold

defined as:

$$TM = \bigcup_{p \in M} T_p M.$$

Naturally, we can always equip a connected manifold with a smooth Riemannian metric g_0 , a smooth $(0, 2)$ tensor field on M . In particular, g_0 is an inner product on $T_p(M)$, for every $p \in M$. Under some special assumptions on our manifold, namely that of geodesic completeness (see e.g. [21]), a smooth Riemannian metric induces a complete metric space structure on M .

Definition 2.2.2. *Let M a smooth, compact, connected Riemannian manifold with boundary. In addition, let $x, y \in M$ fixed, and define $\mathcal{P} = \{\gamma : [0, 1] \rightarrow M, \gamma \in C^1(M), \gamma(0) = x, \gamma(1) = y\}$, the space of C^1 curves connecting x and y . The function $d : M \times M \rightarrow \mathbb{R}$, defined by:*

$$d_{g_0}(x, y) = \inf_{\mathcal{P}} \int_0^1 \sqrt{g(\gamma'(t), \gamma'(t))} dt,$$

is a distance function on M , in the sense that (M, d) satisfies the usual metric space axioms.

Corresponding to Euclidean balls, we can now define geodesic balls on M by the formula:

$$B_r(x) = \{y \in M \mid d_{g_0}(x, y) < r\}.$$

In addition, there are other elementary concepts still valid in our context. One of them is that of a vector field, which we define via an assignment of the form:

$$X : M \ni p \rightarrow X(p) \in T_p M.$$

In local coordinates, we can write:

$$X = \alpha_i \partial_i,$$

for smooth functions $\alpha_i : M \rightarrow \mathbb{R}$, and call X a smooth vector field in that case. The space of all smooth vector fields on M will be denoted by $\Gamma(M)$. Moreover,

we can define a covariant derivative on TM as a map

$$\nabla : \Gamma(M) \times \Gamma(M) \rightarrow \Gamma(M),$$

satisfying the following properties:

1. $\nabla_{fX+gY}Z = f\nabla_XZ + g\nabla_YZ$
2. $\nabla_XfY = X(f)Y + f\nabla_XY$,

for any smooth vector fields X, Y, Z and $f, g \in C^\infty(M)$. This concept extends naturally to $(0, s)$ tensor fields via the following formula:

$$\begin{aligned} \nabla T(X, x_1, \dots, x_s) &= \nabla_X[T(x_1, \dots, x_s)] \\ &\quad - T(\nabla_X x_1, \dots, x_s) - \dots - T(x_1, \dots, \nabla_X x_s). \end{aligned} \quad (2.2.1)$$

Additionally, given smooth vector fields X, Y , one can define a new vector field by considering their Lie bracket, which is the vector field defined by the relation:

$$[X, Y] = XY - YX.$$

The torsion of a connection is then defined by the formula:

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

for $X, Y \in \Gamma(M)$. We can see that the torsion of a connection is a vector field, using the definition of the covariant derivative and that of the Lie bracket. Indeed, $\nabla_X Y - \nabla_Y X$ and $[X, Y]$ are vector fields, hence their difference is a vector field too.

The Levi-Civita connection on M , denoted by ∇^0 , is the unique torsion free connection on TM such that :

$$\nabla_X^0[g_0(Y, Z)] = g_0(\nabla_X^0 Y, Z) + g_0(Y, \nabla_X^0 Z),$$

$\forall X, Y, Z \in TM$ (see [21] for a proof of this existence and uniqueness result). We can then interpret this relation using equation (2.2.1) as the vanishing of the

covariant derivative of the metric tensor. Using the Levi-Civita connection, we can define the Christoffel symbols of g_0 ([5]) as the functions given in local coordinates by:

$$\nabla_i^0 \partial_j = (\Gamma_0)_{ij}^k \partial_k = \frac{1}{2} [\partial_i (g_0)_{lj} + \partial_j (g_0)_{li} - \partial_l (g_0)_{ij}] (g_0)^{kl}. \quad (2.2.2)$$

We will see that we can use the Christoffel symbols to express various notions in local coordinates.

A central concept in the field of Riemannian Geometry is that of curvature. Contrary to the case of surfaces, where we have the Gauss curvature playing a prominent role, on higher dimensional manifolds the several different curvature tensors have different roles. We briefly recall their definition, beginning with that of Riemann curvature.

Definition 2.2.3. *Let X, Y, Z, W smooth vector fields on M . The Riemann curvature of M with respect to g_0 is the $(0,4)$ tensor:*

$$\mathcal{R}_0(X, Y, Z, W) = g_0(W, R_0(X, Y,)Z),$$

where $R_0(X, Y,)Z$ is defined by:

$$R_0(X, Y,)Z = \nabla_X^0 \nabla_Y^0 Z - \nabla_Y^0 \nabla_X^0 Z - \nabla_{[X, Y]}^0 Z.$$

.

In normal coordinates we have the following expression ([5]) for the components of the Riemann curvature:

$$(\mathcal{R}_0)_{kl ij} = \frac{1}{2} ((g_0)_{jk, li} + (g_0)_{il, kj} - (g_0)_{jl, ki} - (g_0)_{ik, lj}).$$

Roughly speaking, the Riemann curvature tensor measures the failure of the covariant derivatives to commute, when we commute their arguments.

The Ricci curvature of M is a symmetric bilinear form, obtained by taking the trace of the Riemann curvature with respect to g_0 . In particular, the Ricci curvature of M with respect to g_0 is the $(0, 2)$ tensor Ric , given in local coordinates by:

$$R_{kl} = g_0^{ij} (\mathcal{R}_0)_{ikjl}.$$

Using the Ricci curvature tensor, we can now define the scalar curvature of our manifold with respect to a given metric. Contrary to the Riemann and Ricci curvature tensors, the scalar curvature is a scalar function defined on a Riemannian manifold. It is obtained by taking the trace of the Ricci tensor of M with respect to g_0 , as we can see from the following :

Definition 2.2.4. *Let (M, g_0) be a smooth, compact Riemannian manifold. The scalar curvature of M with respect to g_0 is the scalar function given in local coordinates by the formula:*

$$R_0 = g_0^{ij} R_{ij}.$$

The presence of a boundary allows us to define further tensors. We begin by introducing a bilinear form on the tangent space to the boundary of our manifold.

Definition 2.2.5. *Let N_0 a unit normal vector field to ∂M , and X, Y smooth vector fields on ∂M . The second fundamental form of \tilde{g}_0 is the symmetric bilinear form defined by:*

$$A_0(X, Y) := \tilde{g}_0(X, \nabla_Y^0 N_0). \quad (2.2.3)$$

Moreover, taking the trace of the second fundamental form we get a scalar function defined on ∂M , its mean curvature. More specifically, it is the smooth function given in local coordinates by the formula:

$$h_0 = \frac{1}{n-1} \tilde{g}^{ij} (A_0)_{ij}.$$

2.3 Some analytical prerequisites

We will now briefly digress, in order to define some analytical objects on manifolds, as well as recall some notions from the theory of elliptic partial differential equations of second order. We begin with the notion of a measure induced by the metric g_0 . In particular, if (M, g_0) is smooth Riemannian manifold, we can always define a measure μ_0 on M relative to metric g_0 :

Definition 2.3.1. *The measure μ_0 induced by g_0 on M is the unique Lebesgue measure defined by integration against the volume form given in local coordinates*

by:

$$\sqrt{|\det g_0|} dx_1 \dots dx_n.$$

Also, note that one can similarly define a surface measure σ_0 with respect to \tilde{g}_0 on ∂M . We will denote by $L^p(M, \mu)$ the usual Lebesgue space of p integrable functions with respect to the measure μ and by $L^\infty(M, \mu)$ the space of essentially bounded, measurable functions on M with respect to μ . We recall that $L^p(M, \mu)$ is equipped with the norm $\|\cdot\|_{L^p(M, \mu)}$, with:

$$\|f\|_{L^p(M, \mu)} = \left(\int_M |f|^p d\mu \right)^{1/p},$$

for a function $f \in L^p(M, \mu)$. Also, the space $L^\infty(M, \mu_0)$ is equipped with the norm:

$$\|\cdot\|_{L^\infty(M, \mu_0)} = \operatorname{ess\,sup}_M |\cdot|.$$

Recall the following basic Lemma, relating the norms of the spaces $L^p(M, \mu_0)$ and $L^\infty(M, \mu_0)$ for a manifold with boundary and finite volume:

Lemma 2.3.2. *Let $f : M \rightarrow \mathbb{R}$ with $f \in L^p(M, \mu_0)$, $1 \leq p \leq \infty$, then*

$$\lim_{p \rightarrow +\infty} \|f\|_{L^p(M, \mu_0)} = \|f\|_{L^\infty(M, \mu_0)}.$$

Proof. Recall that our manifold M compact. Then, it follows that its measure $|M|$ is finite. Using Holder's inequality, we can estimate:

$$\|f\|_{L^p(M, \mu_0)} = \left(\int_M |f|^p d\mu_0 \right)^{1/p} \leq |M|^{\frac{1}{p}} \|f\|_{L^\infty(M, \mu_0)}.$$

Then, letting $p \rightarrow \infty$ we obtain

$$\limsup_{p \rightarrow \infty} \|f\|_{L^p(M, \mu_0)} \leq \|f\|_{L^\infty(M, \mu_0)}. \quad (2.3.1)$$

On the other hand, for $p > 1$, we let $k > 0$ such that $\|f\|_{L^\infty(M, \mu_0)} > k$ and define:

$$A_k = \{x \in M, |f(x)| > k\}.$$

Note that, under our assumptions, we have $|A_k| > 0$. Then, it follows that:

$$(\int_M |f|^p d\mu_0)^{1/p} \geq (\int_{A_k} |f|^p d\mu_0)^{1/p} \geq k|A_k|^{1/p},$$

implying that:

$$\liminf_{p \rightarrow +\infty} \|f\|_{L^p(M, \mu_0)} \geq k,$$

for $k \leq \|f\|_{L^\infty(M, \mu_0)}$. Combining this with (2.3.1) we are done. □

This Lemma will be essential to our later considerations. It will allow us to approximate our minimization problem in L^∞ by the limit of the corresponding p -problem. Now, using the definition of the L^p spaces, we can define the Sobolev spaces $W^{k,p}$ on a manifold with boundary, as usual. A basic reference for Sobolev spaces in general is [2] and [5], [20] in the geometric context.

Definition 2.3.3. *Let $k \in \mathbb{N}$ and $p \geq 1$ a real number and $u : M \rightarrow \mathbb{R}$ a smooth function, and set*

$$\|u\|_{k,p} = \sum_{j=0}^k (\int_M |\nabla^j u|^p d\mu_0)^{1/p}.$$

The Sobolev space $W^{k,p}(M, g_0)$ is defined as the completion of the space $C^\infty(\overline{M}, g_0)$, with respect to the norm $\|\cdot\|_{W^{k,p}(M, g_0)}$.

In addition to Sobolev spaces on a bounded domain and on our manifold, we will need to work with the following variant of them:

Definition 2.3.4. *The homogeneous Sobolev space D_1^2 is the vector space defined as the completion of the space $C_0^\infty(\mathbb{R}^n)$, with respect to the norm:*

$$\|\cdot\|_{D_1^2(\mathbb{R}^n)} = \|\nabla \cdot\|_{L^2(\mathbb{R}^n)}.$$

The basic tool that allows us to relate the Lebesgue and Sobolev spaces on a bounded domain is the Sobolev Embedding Theorem. We present here a version from [31] fitted to our context:

Theorem 2.3.5. *Let M a compact manifold with smooth boundary and $1 \leq p \leq +\infty$. If $kp < n$, we have:*

$$W^{k,p}(M, g_0) \hookrightarrow L^q(M, \mu_0),$$

continuously for $q \leq \frac{np}{n-kp}$, and compactly for $q < \frac{np}{n-kp}$.

If $0 \leq k - \frac{n}{p} < m + 1$, we have:

$$W^{k,p}(M, g_0) \hookrightarrow C^{m,\alpha}(M, g_0),$$

continuously for $\alpha \leq k - m - \frac{n}{p}$, and compactly for $\alpha < k - m - \frac{n}{p}$.

We remark that when $k = 1$ compactness fails for $q = \frac{2n}{n-2} = 2^*$, the so-called critical exponent. This lack of compactness is reflected in several results, ranging from nonexistence results for certain partial differential equations, like the Pohozaev result (see [31]), to problems related to geometry like the Yamabe problem. We will attempt to illustrate the effect of the critical exponent in a geometric context, later on in this chapter.

A concept closely related to the Sobolev Embedding Theorem, which is important for some of our later considerations, is that of Euclidean Sobolev Inequalities and their best constants. In general, given normed vector spaces $(A, \|\cdot\|_A)$, $(B, \|\cdot\|_B)$ and a continuous embedding $A \hookrightarrow B$, we want to determine the smallest constant C and corresponding vectors f_{ext} , for which the following inequality holds:

$$\|f_{ext}\|_B \leq C \|f_{ext}\|_A.$$

In what follows, we recall the classical Euclidean Sobolev Inequality [4], [32].

Theorem 2.3.6. ([4],[32]) *The smallest positive constant K_n for which the following inequality holds*

$$\left(\int_{\mathbb{R}^n} \phi^{2^*} dx \right)^{1/2^*} \leq K_n \left(\int_{\mathbb{R}^n} |\nabla \phi|^2 dx \right)^{1/2},$$

for every $\phi \in W^{1,2}(\mathbb{R}^n)$, is given by the expression:

$$K_n = \sqrt{\frac{4}{n(n-2)\omega_n^{2/n}}}, \quad (2.3.2)$$

where ω_n stands for the volume of the unit sphere S^n .

For a proof of the Theorem we refer to [5]. The sharp constant K_n for this inequality was explicitly calculated in [4] and, independently, in [32] by Aubin and Talenti respectively. Moreover, it is known that the sharp constant is attained by functions of the form:

$$\phi_\lambda(x) = (\lambda + |x|^2)^{1-\frac{n}{2}}, \quad (2.3.3)$$

for $\lambda \in \mathbb{R}^+$. In addition, we can translate those solutions for $\alpha \in \mathbb{R}^n$ and get new solutions of the form:

$$\phi_{\alpha,\lambda}(x) = (\lambda + |x - \alpha|^2)^{1-\frac{n}{2}}.$$

These function have some interesting properties, as far as their regularity is concerned, and we will try to emulate their behavior later on this text.

Having given the necessary definitions regarding function spaces, we can now proceed to discussing notions related to second order elliptic equations and the regularity of their solutions. We begin by giving the definition of an elliptic operator of second order in our context. The standard reference for elliptic equations of second order is [16].

Definition 2.3.7. *Let M a smooth compact manifold with smooth boundary and $\Omega \subset \mathbb{R}^n$ open, such that Ω belongs to an atlas of M . A linear differential operator of second order is an operator given in local coordinates by the expression:*

$$L = \partial_i(a_{ij}\partial_j) + b_i\partial_i + c, \quad (2.3.4)$$

for $a_{ij}, b_i, c \in L^\infty(M)$. The operator L is elliptic, if $\forall p \in M$ there exists $\lambda = \lambda(p) > 0$, such that $\forall \xi \in \mathbb{R}^n$ we have:

$$\lambda^{-1}|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \lambda|\xi|^2.$$

The operator L is uniformly elliptic if (2.3.4) holds with λ independent of p .

The prototype operator we will use, is a perturbation of the Laplace-Beltrami operator, which itself is a generalization of the ordinary Laplacian.

Definition 2.3.8. *Let (M, g_0) a Riemannian manifold with boundary, the Laplace-Beltrami operator on M is the second order differential operator acting on functions, given in local coordinates by:*

$$\Delta_{g_0} = \frac{1}{\sqrt{|g_0|}} \partial_i (g_0^{ij} \sqrt{|g_0|} \partial_j). \quad (2.3.5)$$

For more details on other generalizations of the Laplacian to manifolds, we refer to [21]. A particular aspect of elliptic operators is that of the regularity of their solutions. Roughly speaking, this can be described in the following naive way: Let L an elliptic operator and u a solution of:

$$Lu = w.$$

If w obeys some good regularity properties, then so does the solution u , but in a different sense. In a more detailed way, we will work with the so-called Schauder and L^p theories. In the Schauder theory, the central idea is that we can recover Holder continuity of u up to its second derivatives, under the assumption that w is Holder continuous. On the other hand, if w belongs to some L^p space, with $p > 1$, we can deduce that u has weak derivatives in L^p up to second order.

Let us state the following version, from [31], of the Schauder theory that is needed for this work.

Theorem 2.3.9. *Let $\Omega \subset \mathbb{R}^n$ a $C^{2,\alpha}$ bounded domain, L an elliptic operator of the form (2.3.4) with coefficients in $C^{0,\alpha}(\Omega)$, and u a solution of the equation:*

$$Lu = w,$$

with $w \in C^{0,\alpha}(\overline{\Omega})$ and $u = u_0 \in C^{2,\alpha}(\overline{\Omega})$ over the boundary. Then, $u \in C^{2,\alpha}(\overline{\Omega})$ and in addition

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(\|w\|_{C^{0,\alpha}(\overline{\Omega})} + \|u\|_{C^\infty(\Omega)} + \|u_0\|_{C^{2,\alpha}(\overline{\Omega})}).$$

If the righthandside w satisfies some weaker regularity assumptions, then u belongs to some Sobolev space. We will now state a version of a relevant Theorem of that form from [16].

Theorem 2.3.10. *Let $\Omega \subset \mathbb{R}^n$ a $C^{1,1}$ bounded domain, L an elliptic operator of the form (2.3.4), and $u \in W_{loc}^{2,p}(\Omega, g_0) \cap L^p(\Omega, \mu_0)$ a solution of the equation*

$$Lu = f$$

in Ω , with $f \in L^p(\Omega, \mu_0)$. In addition, suppose that the coefficients of L satisfy

1. $\alpha_{ij} \in C^0(\overline{\Omega})$,
2. $b_i, c \in L^\infty(\Omega)$,

and that:

$$u - u_0 \in W_0^{1,p}(\Omega),$$

for a $u_0 \in W^{2,p}(\Omega)$. Then, we have the following estimate:

$$\|u\|_{W^{2,p}(\Omega)} \leq C[\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \|u_0\|_{W^{2,p}(\Omega)}], \quad (2.3.6)$$

with the constant $C = C(n, p, \Lambda, \Omega)$.

Under some stronger assumptions on the coefficients of the operator L , some better regularity results and estimates are possible, as we can see from the following version of a result from [16].

Theorem 2.3.11. *Let M a smooth compact manifold, L a strictly elliptic operator of the form (2.3.4), with the coefficients of L satisfying the following:*

1. $\alpha_{ij} \in C^0(\overline{M})$,
2. $b_i, c \in L^\infty(M)$,
3. $c \leq 0$.

If $f \in L^p(M)$, with $1 < p < \infty$, then the Dirichlet problem

$$Lu = f, \quad \text{in } M$$

$$u = 0 \quad \text{on } \partial M,$$

admits a unique solution $u \in W^{2,p}(M)$. In addition, we have the following estimate:

$$\|u\|_{W^{2,p}(M)} \leq C[\|Lu\|_{L^p(M)}], \quad (2.3.7)$$

with the constant $C = C(n, p, \Lambda, M)$.

In addition to the regularity obtained by estimating as above, the usual tactic is to combine estimates like (2.3.7), with the Sobolev Embedding Theorem 2.3.5. Then additional regularity is gained, provided certain exponents are large enough, and Schauder or L^p estimates might be implemented once more, leading to a bootstrapping argument.

We conclude this section, with a version of a Harnack Inequality due to Han-Lin [18], as presented in [11]. Harnack inequalities are common results in the theory of elliptic equations, beginning from the Harnack inequality for the Laplace equation. We will use the following lemma later on the text, to prove that certain quantities do not vanish.

Lemma 2.3.12. *Let $B_0(3\delta)$, be the ball in \mathbb{R}^n of center 0 and radius 3δ , and let g be a Riemannian metric on $B_0(3\delta)$. Let $A > 0$ be such that for any smooth function ϕ with compact support in $B_0(3\delta)$,*

$$\|\phi\|_{L^{2^*}(B_{2\delta}(0))} \leq A\|\nabla\phi\|_{L^2(B_{2\delta}(0))},$$

where the norms are taken with respect to g , and $u \in C^1(B_0(2\delta))$, $u > 0$, be such that

$$-\Delta_g u \leq fu$$

where $\int_{B_{2\delta}(0)} |f|^r d\mu_g \leq K$ for some $r > n/2$. Then, for all $p > 0$,

$$\sup_{B_\delta(0)} u \leq C\|u\|_{L^p(B_{2\delta}(0))},$$

where the above norm is taken with respect to g , and $C = C(n, A, K, p, r, \delta)$ depends only on n, A, K, p, r , and δ .

2.4 Conformal changes of metric and transformation laws

In this section we will study how certain geometric quantities related to a fixed Riemannian metric g_0 transform under a conformal change of metric. Firstly, we prove a transformation law for the Christoffel symbols of a Levi-Civita connection.

Lemma 2.4.1. *Let $f : M \rightarrow \mathbb{R}$ a smooth function, and $g = e^{2f}g_0$ a metric conformal to g_0 . The transformation law for the Christoffel symbols after a conformal change is given in local coordinates by the formula:*

$$\Gamma_{ij}^l = [(\partial_i f)\delta_j^l + (\partial_j f)\delta_i^l - (g_0)^{kl}(\partial_k f)(g_0)_{ij}] + (\Gamma_0)_{ij}^l. \quad (2.4.1)$$

Proof. After taking into account that the Christoffel symbols with respect to g_0 are given by:

$$(\Gamma_0)_{ij}^l = \frac{1}{2}[\partial_i(g_0)_{kj} + \partial_j(g_0)_{ki} - \partial_k(g_0)_{ij}](g_0)^{kl},$$

we have:

$$\begin{aligned} \Gamma_{ij}^l &= \frac{1}{2}[\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}]g^{kl} \\ &= [(\partial_i f)(g_0)_{kj} + (\partial_j f)(g_0)_{ki} - (\partial_k f)(g_0)_{ij}](g_0)^{kl} + \frac{1}{2}[e^{2f}(\partial_i(g_0)_{kj} + \partial_j(g_0)_{ki} - \partial_k(g_0)_{ij})]g^{kl} \\ &= [(\partial_i f)(g_0)_{kj} + (\partial_j f)(g_0)_{ki} - (\partial_k f)(g_0)_{ij}](g_0)^{kl} + \frac{1}{2}[\partial_i(g_0)_{kj} + \partial_j(g_0)_{ki} \\ &\quad - \partial_k(g_0)_{ij}](g_0)^{kl} = [(\partial_i f)(g_0)_{kj} + (\partial_j f)(g_0)_{ki} - (\partial_k f)(g_0)_{ij}](g_0)^{kl} + (\Gamma_0)_{ij}^l \\ &= [(\partial_i f)\delta_j^l + (\partial_j f)\delta_i^l - (g_0)^{kl}(\partial_k f)(g_0)_{ij}] + (\Gamma_0)_{ij}^l. \end{aligned}$$

□

Naturally, we want to know how the volume of a manifold behaves under a conformal change of metric. In that direction, we now give the corresponding transformation law.

Lemma 2.4.2 (Transformation law for the measure). *Let (M, g_0) be a smooth, compact Riemannian manifold with volume form μ_0 and $f : M \rightarrow \mathbb{R}$ a smooth function on M . If $g = f g_0$ is a conformal metric to g_0 , then the volume forms of*

g and g_0 are related by:

$$\mu = f^{n/2} \mu_0.$$

In particular, for a smooth, positive function $u : M \rightarrow \mathbb{R}$, the conformal metric $g = u^{2^*-2} g_0$ has volume form:

$$\mu = u^{2^*} \mu_0. \quad (2.4.2)$$

Proof. Writing $g_0 = g_{ij} dx^i \otimes dx^j$ in local coordinates, we have:

$$\mu_0 = \sqrt{|\det g_0|} dx^1 \dots dx^n \quad (2.4.3)$$

$$g = f g_{ij} dx^i \otimes dx^j$$

and

$$\begin{aligned} \mu &= \sqrt{|\det g|} dx^1 \dots dx^n = \sqrt{|[\det f I_n] \det g_{ij}|} dx^1 \dots dx^n \\ &= \sqrt{f^n} \sqrt{|\det g|} dx^1 \dots dx^n = f^{n/2} \mu_0. \end{aligned}$$

Taking into account the fact that $2^* - 2 = \frac{4}{n-2}$, the result follows. \square

We continue by giving a transformation law for the scalar curvature of our manifold after a conformal change. The resulting transformation law is one of the main objects of study in this work.

Lemma 2.4.3 (Transformation law for the scalar curvature). *Let $u : M \rightarrow \mathbb{R}$, a smooth positive function. If $g = u^{2^*-2} g_0$ is a conformal metric to g_0 , then we have:*

$$-c_n \Delta_{g_0} u + R_0 u = R u^{2^*-1}. \quad (2.4.4)$$

Proof. Let $f : M \rightarrow \mathbb{R}$ a smooth function, such that $e^{2f} = u^{2^*-2}$. Writing $g = e^{2f} g_0$, we have:

$$R = e^{-2f} (R_0 - 2(n-1) \Delta_{g_0} f - (n-1)(n-2) |\nabla f|^2), \quad (2.4.5)$$

see [7]. It follows that $e^{-2f} = u^{2-2^*}$, and that

$$f = \frac{2^* - 2}{2} \ln u. \quad (2.4.6)$$

Now taking into account that:

$$\nabla f = \frac{2^* - 2}{2} \frac{\nabla u}{u},$$

we have the following expression for the Laplacian of f , with respect to g_0 :

$$\Delta_{g_0} f = \frac{2^* - 2}{2} \frac{(\Delta_{g_0} u)u - |\nabla u|^2}{u^2}.$$

Hence, we can rewrite equation (2.4.5) in terms of u as:

$$R = u^{2^*-2} \left(R_0 - \frac{8(n-1)}{2(n-2)} \left(\frac{u \Delta_{g_0} u - |\nabla u|^2}{u^2} \right) - \frac{4(n-1)(n-2)}{(n-2)^2} \frac{|\nabla u|^2}{u^2} \right),$$

which in turn implies that:

$$Ru^{2^*-2} = \left(R_0 - \frac{4(n-1)}{(n-2)} \frac{\Delta_{g_0} u}{u} + \frac{4(n-1)}{(n-2)} \frac{|\nabla u|^2}{u^2} - \frac{4(n-1)}{(n-2)} \frac{|\nabla u|^2}{u^2} \right),$$

finishing our proof. □

We now proceed to study the transformation law of curvature tensors related to the boundary. Initially, we will study how the second fundamental form of ∂M transforms under a conformal change of metric.

Lemma 2.4.4 (Transformation law for the mean curvature). *Let $f : M \rightarrow \mathbb{R}$ a smooth function on M . The second fundamental form A with respect to the metric $\tilde{g} = e^{2f} \tilde{g}_0$, is given by:*

$$A(X, Y) = g(X, \nabla_Y N) = e^f [g_0(X, \nabla_Y^0 N) + g_0(X, Y) \frac{\partial f}{\partial \nu_0}], \quad (2.4.7)$$

for all $X, Y \in \Gamma(\partial M)$, where N is a unit normal vector field to ∂M with respect to g .

Proof. If N_0 is a unit normal vector field for g_0 , then the renormalized vector field N , defined by $N = \frac{N_0}{e^f}$ is a unit vector field for g , since $g(N, N) = 1$. The conformal metric g induces a Levi-Civita connection obeying (2.4.1). Moreover,

the associated second fundamental form satisfies:

$$\begin{aligned} A(X, Y) &= g(X, \nabla_Y N) = e^f g_0(X, \nabla_Y N_0) = e^f g_0(X, \nabla_Y^0 N_0) \\ &+ e^f g_0(X, Y(f)N_0) + e^f g_0(X, N_0(f)Y) - g_0(Y, N_0)e^f g_0(X, \text{grad}f), \end{aligned}$$

following (2.4.1). Then, taking into account the fact that N_0 is normal to X, Y we are done. \square

The following Lemma gives a transformation law for the mean curvature after a conformal change, and can be thought of as a boundary analogue of Lemma 2.4.3.

Lemma 2.4.5. *Let u a smooth positive function on M , and $g = u^{2^*-2}g_0$ a metric conformal to g_0 . If h_g is the mean curvature of the metric g , then the following equation holds on ∂M :*

$$\frac{n-2}{2}h_g u^{\frac{n}{n-2}} = \frac{n-2}{2}h_0 u + \frac{\partial u}{\partial \nu_0}. \quad (2.4.8)$$

Proof. Let $f : \overline{M} \rightarrow \mathbb{R}$ be a smooth function, such that:

$$g = e^{2f}g_0 = u^{\frac{4}{n-2}}g_0.$$

From equation (2.4.7), we know that in local coordinates the components of the new second fundamental form A transform as:

$$A_{ij} = e^f (A_0)_{ij} + (\tilde{g}_0)_{ij} e^f \frac{\partial}{\partial \nu_0} f.$$

Taking into account the definition of the mean curvature, and taking traces with respect to the conformal metric g , we have:

$$h_g = \frac{1}{n-1} \tilde{g}^{ij} A_{ij} = \frac{1}{n-1} e^{-2f} (\tilde{g}_0)^{ij} A_{ij} = e^{-f} h_0 + e^{-f} \frac{\partial}{\partial \nu_0} f.$$

Finally, recaling the definition of f , we are done. \square

Equation (2.4.8) will be used as a nonlinear boundary condition on some of our

later considerations.

In order to conclude this part of our work, we state the following transformation law for the Laplace-Beltrami operator after a conformal change.

Lemma 2.4.6 (The Laplacian of a conformal metric). *Let $g = u^{2^*-2}g_0$ a metric conformal to g_0 , for a smooth positive function u on M . Then, if Δ_g is the Laplacian of the metric g and $f \in C^\infty(M)$, we have:*

$$\Delta_g f = \frac{u^{-2^*}}{\sqrt{|g_0|}} \partial_j (g_0^{ij} u^2 \sqrt{|g_0|} \partial_i f). \quad (2.4.9)$$

Proof. Using equation (2.3.5), we know that in local coordinates the Laplacian is given by:

$$\Delta_{g_0} f = \frac{1}{\sqrt{|g_0|}} \partial_j (g_0^{ij} \sqrt{|g_0|} \partial_i f).$$

The components of the inverse of g are trivially given by $g^{ij} = g_0^{ij} u^{2-2^*}$, in local coordinates. In addition, the volume element transforms by $\sqrt{|g|} = (u^{2^*-2})^{n/2} \sqrt{|g_0|} = u^{2^*} \sqrt{|g_0|}$, by using equation (2.4.2). So, we get

$$\begin{aligned} \Delta_g f &= \frac{1}{\sqrt{|g|}} \partial_j (g^{ij} \sqrt{|g|} \partial_i f) \\ &= \frac{(u^{2-2^*})^{n/2}}{\sqrt{|g_0|}} \partial_j (g_0^{ij} u^{2-2^*} (u^{2^*-2})^{n/2} \sqrt{|g_0|} \partial_i f) \\ &= \frac{(u^{-2^*})}{\sqrt{|g_0|}} \partial_j (g_0^{ij} u^2 \sqrt{|g_0|} \partial_i f). \end{aligned}$$

□

2.5 Conformally Covariant Operators

The starting point of this section is the notion of bidegree of a metrically defined operator, following the definition by S.Y. A. Chang in [9].

Definition 2.5.1. *A metrically defined operator A_g is conformally covariant of bidegree (a, b) , if after a conformal change of metric of the form $g = e^{2w} g_0$, where*

w is a smooth function on M , we have:

$$A_g(f) = e^{-bw} A_{g_0}(e^{aw} f). \quad (2.5.1)$$

For $n = 2$, the Laplacian Δ_g is conformally covariant of bidegree (0,2). Indeed, let (S, g_0) a smooth Riemannian surface and g is a metric conformal to g_0 , of the same form as above. Then, by the transformation law (2.4.2) for the volume element and the equation (2.3.5) defining the Laplace operator, we get:

$$\begin{aligned} \Delta_g f &= \frac{1}{\sqrt{|g|}} \partial_j (g^{ij} \sqrt{|g|} \partial_i f) = e^{-2w} \frac{1}{\sqrt{|g_0|}} \partial_j (e^{-2w} g^{ij} \sqrt{|g|} e^{2w} \partial_i f) = \\ &= e^{-2w} \Delta_{g_0} f. \end{aligned}$$

Nevertheless, the situation is not similar in the case $n \geq 3$, as can be seen from Lemma 2.4.6, hence we need to substitute the usual Laplace-Beltrami operator by the conformal Laplacian.

Definition 2.5.2. *The conformal Laplacian of g_0 is the second order differential operator L_{g_0} , defined by:*

$$L_{g_0} = -c_n \Delta_{g_0} + R_0,$$

where c_n is defined by $c_n = 4 \frac{n-1}{n-2}$.

Using the conformal Laplacian, the transformation law (2.4.4) for the scalar curvature R_0 , under a conformal change of metric $g = u^{2^*-2} g_0$, can now be written as:

$$L_{g_0} u = R u^{2^*-1}.$$

Combining the transformation laws for the measure (2.4.2), the Laplacian (2.4.9) and the scalar curvature (2.4.4), we have the following useful relation on the conformal Laplacian of a background metric, and that after a conformal change.

Lemma 2.5.3. *Let $g = u^{2^*-2} g_0$ a conformal metric to g_0 , with $u \in C^\infty(M)$ positive and $\phi \in C^\infty(M)$, then:*

$$u^{1-2^*} L_{g_0}(u\phi) = L_g \phi. \quad (2.5.2)$$

Proof. Using the product rule, (2.4.4) and (2.4.9), we have:

$$u^{1-2^*} L_{g_0}(u\phi) = u^{1-2^*} [-c_n \Delta_{g_0}(u\phi) + R_0 u\phi] = R\phi - c_n \Delta_g \phi,$$

following equation (2.4.9). □

We note that by substituting $\phi = 1$ in the above, we recover the transformation law (2.4.4) for the scalar curvature. So, we are now in the position to justify our discussion of conformally covariant operators, by proving the following:

Lemma 2.5.4. *The conformal Laplacian L_g is a conformally covariant operator of bidegree $(\frac{n-2}{2}, \frac{n+2}{2})$, for every $n \in \mathbb{N}$, with $n \geq 3$.*

Proof. Let $\phi \in C^\infty(M)$. Taking into account equation (2.5.2), and letting

$$u^{2^*-2} = e^{2w},$$

for a smooth $w : M \rightarrow \mathbb{R}$, we trivially have:

$$u^{1-2^*} L_{g_0} \phi = L_g(u^{-1} \phi).$$

That fact then, implies that:

$$e^{\frac{n+2}{2}w} L_{g_0} \phi = L_g(e^{\frac{n-2}{2}w} \phi),$$

thus concluding our proof. □

A boundary counterpart of the conformal Laplacian can also be defined, in a similar way. For a Riemannian metric g , let B_g be the operator acting on smooth functions $\phi : \partial M \rightarrow \mathbb{R}$, given by the formula:

$$B_g(\phi) = \frac{\partial \phi}{\partial \nu} + \frac{n-2}{2} h_g \phi.$$

From the transformation law (2.4.8), we can deduce that:

$$B_g(\phi) = u^{\frac{n}{2-n}} B_{g_0}(\phi u).$$

Moreover, we have the following:

Lemma 2.5.5. *The operator $B_g : \partial M \rightarrow \mathbb{R}$ is a conformally covariant operator of bidegree $(\frac{n}{2}, \frac{n-2}{2})$, for every $n \in \mathbb{N}$, with $n \geq 3$.*

Proof. Let $\phi \in C^\infty(\overline{M})$. If $w : \overline{M} \rightarrow \mathbb{R}$ is a smooth function, such that:

$$e^{2w} = u^{2^*-2},$$

then trivially:

$$e^{\frac{nw}{2}} = u^{\frac{n}{n-2}},$$

and

$$e^{\frac{(n-2)w}{2}} = u.$$

Moreover, after taking into account the transformation law (2.4.8), we have:

$$\begin{aligned} u^{-\frac{n}{n-2}} B_{g_0}(\phi u) &= u^{-\frac{n}{n-2}} \left(\frac{\partial u}{\partial \nu_0} \phi + \frac{\partial \phi}{\partial \nu_0} u + \frac{n-2}{2} h_{g_0} \phi u \right) \\ &= u^{-\frac{n}{n-2}} \left(\frac{\partial u}{\partial \nu_0} + \frac{n-2}{2} h_{g_0} u \right) \phi + u^{-\frac{n}{n-2}} \frac{\partial \phi}{\partial \nu_0} u \\ &= \frac{n-2}{2} h_g \phi + \frac{\partial \phi}{\partial \nu} = B_g \phi, \end{aligned}$$

thus concluding our proof. □

2.6 Preliminaries from Geometric Measure Theory

In this section we discuss some notions from Geometric Measure Theory, which will be needed in the sequel. A basic reference for this section is [30].

Definition 2.6.1. *The set $\Omega \subset \mathbb{R}^{n+k}$ is a countably n -rectifiable set, if there exists a family $\{N_i\}_{i \in \mathbb{N}}$, with $N_i \subset \mathbb{R}^{n+k}$ and Lipschitz maps $f_i : \mathbb{R}^n \rightarrow \mathbb{R}^{n+k}$, such that*

$$\Omega \subset N_0 \cup \bigcup_i N_i,$$

where $\mathcal{H}^n(N_0) = 0$ and:

$$N_i = f_i(\mathbb{R}^n),$$

$\forall i \in \mathbb{N}, i \neq 0$.

For the sake of completeness, we also give the following Lemma from [30], providing a simpler characterization of countably n -rectifiable sets.

Lemma 2.6.2. *The set $\Omega \subset \mathbb{R}^{n+k}$ is a countably n -rectifiable set, iff $M \subset N_0 \cup \bigcup_i N_i$, where $\mathcal{H}^n(N_0) = 0$, and N_i are n -dimensional C^1 embedded submanifolds of \mathbb{R}^{n+k} , $\forall i \in \mathbb{N}^*$.*

2.7 The Direct Method and Critical Nonlinearities

In order to conclude our set of preliminary material for this work, we give a brief discussion of the Direct Method, which is a typical approach to attack a variational problem. In this section we give the version that we will use, in order to prove existence for a family of problems that will approximate our minimization problem. A basic and complete reference for this topic is [31].

Theorem 2.7.1. *Suppose $(X, \|\cdot\|_X)$ is a reflexive Banach space, and let $C \subset X$ be weakly closed. Let $E : M \rightarrow \mathbb{R}$ be a functional on M , such that:*

1. *E is (sequentially) weakly lower semicontinuous on C with respect to X and*
2. *every sequence $\{u_m\} \subset C$, with $E(u_m) \rightarrow \inf_M E(u)$, is bounded in X .*

Then E is bounded below and attains its infimum in C .

In order to demonstrate some of the advantages and the disadvantages of this approach in our context, we will now give an overview of the variational formulation of the Yamabe problem. In that framework $X = W^{1,2}(M, g_0)$, for a smooth compact and closed manifold (M, g_0) . We can then try to cast the problem of finding a constant scalar curvature metric within a fixed conformal class in a variational manner, beginning with specifying the functional we will attempt to minimize.

In that direction, we let $E_Y : W^{1,2}(M, g_0) \rightarrow \mathbb{R}$, and $V_Y : L^{2^*}(M, \mu_0) \rightarrow \mathbb{R}$ being defined by:

$$E_Y(u) = \frac{1}{2}(c_n \int_M |\nabla u|^2 d\mu_0 + \int_M R_0 u^2 d\mu_0), \quad (2.7.1)$$

and

$$V_Y(u) = \frac{1}{2^*} \int_M u^{2^*} d\mu_0.$$

We note that the functional E_Y is known as the Yamabe energy in that context [29].

We then consider the following minimization problem:

$$\min_{u \in W^{1,2}(M, g_0) \cap M_1} E_Y(u),$$

where $M_1 = \{u \in W^{1,2}(M, g_0), V_Y(u) = 1\}$. We remark, once more, that compactness of the Sobolev Embedding $W^{1,2} \hookrightarrow L^q(\Omega)$ fails for $q = \frac{2n}{n-2} = 2^*$.

Initially, we give a brief description of problems that arise from this lack of compactness. Suppose that we are given a bounded sequence $\{u_m\} \subset W^{1,2}(M, g_0)$. Since $W^{1,2}(M, g_0)$ is a reflexive space, we can deduce that up to a subsequence, which we still denote by u_m , we have

$$u_m \rightharpoonup u \text{ in } W^{1,2}(M, g_0),$$

for a limit function u . Additionally, the compactness of the Sobolev Embedding Theorem 2.3.5 for $q < 2^*$ implies that, up to a subsequence,

$$u_m \rightarrow u \text{ in } L^q(M, \mu_0),$$

for $q < 2^*$. Nevertheless, we cannot deduce that $u_m \rightarrow u$ in L^{2^*} strongly and this is one of the main problems, when we have to solve variational problems involving the critical exponent.

If we calculate the Euler - Lagrange equation of the aforementioned minimization problem (see [23]), with corresponding Lagrange multiplier c , we get the

following equation:

$$-c_n \Delta_{g_0} u + R_0 u = c u^{2^*-1}.$$

Thus, if u is a minimizer of our problem, then the metric $g = u^{2^*-2} g_0$ has constant scalar curvature. If we try to apply the Direct Method for this problem, we will encounter certain difficulties. Indeed, letting $\{u_m\} \subset W^{1,2}(M, g_0)$ a minimizing sequence for $E_Y(u)$ with $V_Y(u_m) = 1$, then since $E_Y(u_m) \rightarrow \inf E_Y(u)$:

$$\left(\int_M |c_n \nabla u_m|^2 d\mu_0 + \int_M R_0 u_m^2 d\mu_0 \right) \leq C.$$

Taking advantage of our constraint and Holder's inequality, we have:

$$\left(\int_M u_m^2 d\mu_0 \right) \leq C \left(\int_M u_m^{2^*} d\mu_0 \right) = C. \quad (2.7.2)$$

Hence :

$$\left(\int_M |c_n \nabla u_m|^2 d\mu_0 \right) \leq C - \int_M R_0 u_m^2 d\mu_0 \leq C + \max_M |R_0| \int_M u_m^2 d\mu_0, \quad (2.7.3)$$

thus $\left(\int_M |\nabla u_m|^2 d\mu_0 \right)$ is bounded. Combining equations (2.7.2) and (2.7.3), we deduce that $\{u_m\}$ is bounded in $W^{1,2}(M, g_0)$. Then, as we have already remarked

- $u_m \rightharpoonup u$ in $W^{1,2}(M, g_0)$,
- $u_m \rightarrow u$ in $L^q(M, \mu_0)$,

for $q < 2^*$, up to a subsequence. Since u_m is bounded in L^{2^*} , we can also deduce that $u_m \rightharpoonup u$ in $L^{2^*}(M, \mu_0)$ too. Nevertheless, this is not enough to conclude, since the fact that $V_Y(u) = 1$ is not necessarily true.

A natural remedy, for a situation like that, can be found by considering bounds on the infimum we want to achieve. Namely, we restrict our attention to conditions like:

$$\inf_{u \in W^{1,2}(M, g_0) \cap M_1} E_Y(u) < s_* = \inf_{u \in W^{1,2}(S^n, g_s) \cap M_1} E_Y(u),$$

where g_s is the standard metric on S^n . We will see that, in order to obtain solutions to our minimization problem and face the corresponding lack of compactness

due to the presence of the critical exponent, we will impose some bounds on the corresponding energy functional.

A similar approach was used by Escobar [13], [14], in the case of manifolds with boundary. In that case, there exist two variants of the Yamabe problem, as it was discussed in the Introduction. For the sake of simplicity, and since we will need this later on, we now give an overview of only one of those generalizations loosely following [13].

In that direction, we define:

$$G(u) = \frac{\int_M |\nabla u|^2 d\mu_0 + \frac{1}{c_n} \int_M R_0 u^2 d\mu_0 + \frac{(n-2)}{2} \int_{\partial M} h_0 u^2 d\sigma_0}{\left(\int_{\partial M} u^{\frac{2(n-1)}{(n-2)}} d\sigma_0 \right)^{\frac{n-2}{n-1}}},$$

following Escobar, for $u \in C^\infty(M, g_0)$, $u \neq 0$. If we denote the minimizers of G by u_G , we get corresponding metrics $\bar{g} = u_G^{2^*-2} g_0$, with $R_{\bar{g}} = 0$ and $h_{\bar{g}} = c$. In that case, a minimizer for $\|\cdot\|_\infty$ of the scalar curvature within the conformal class of g_0 is given by \bar{g} . Moreover, the Euler-Lagrange equations corresponding to this minimization problem are given by:

$$-c_n \Delta_{g_0} u + R_0 u = 0, \quad \text{in } M$$

$$\frac{\partial u}{\partial \nu_0} + \frac{n-2}{2} h_0 u = c u^{\frac{n}{n-2}}, \quad \text{on } \partial M.$$

We note the presence of a nonlinear boundary condition, which is the source of similar problems to the closed manifold case. Namely, ([13]) the exponent $\frac{2(n-1)}{n-2}$ is critical for the Trace Sobolev Embedding $W^{1,2}(M) \hookrightarrow L^q(\partial M)$. Thus, it is reasonable to expect some complications arising. Contrary to that, we will study a boundary value problem under Dirichlet data, only controlling integral means of terms involving h_g . It is then natural to expect some weaker results in our case, something which is true, as it will become apparent in the sequel.

CHAPTER 3

STATEMENT OF THE MAIN RESULT

In this chapter we give the statement of our problem, along with all the necessary framework in order to cast it in a variational manner. We then give the statement of our main result, along with some comments and an outline of the strategy for its proof.

3.1 Statement of the problem

Let (M, g_0) be a smooth compact Riemannian manifold with smooth boundary, and $u : M \rightarrow \mathbb{R}$ a smooth positive function. As we have already seen, if g_0 is a Riemannian metric on M , we can always consider a metric conformally related to it, defined by:

$$g = u^{2^*-2} g_0,$$

when the dimension of M satisfies $n \geq 3$. Moreover, from the transformation law (2.4.4), we know that the scalar curvature R of the metric g is a functional of u :

$$R = (-c_n \Delta_{g_0} u + R_0 u) u^{1-2^*}.$$

The question that we attempt to answer is the following:

Is there a metric in the conformal class of g_0 , with scalar curvature R minimizing

the L^∞ norm?

Rigorously written, this amounts to the existence of a minimizer of the functional :

$$E(u) = \|R\|_{L^\infty(M)}.$$

If we write $R = R_g$ for the moment, in order to stress the dependence on g , we ask whether the following infimum is attained:

$$\inf_{g \in [g_0]} \|R_g\|_{L^\infty(M)},$$

where $[g_0]$ is the class of metrics pointwise conformally equivalent to g_0 . As we have already stressed, a standard way to treat variational problems is the use of the Direct Method Theorem 2.7.1. A disadvantage of that approach in our case is that the space $L^\infty(M, \mu_0)$ is not reflexive (see e.g. [8]). Thus, we cannot even establish existence of a minimizer in that way, however restrictive we are. Nevertheless, this complication can be treated by an approximation scheme. Namely, we can first attempt to minimize the functional:

$$E_p(u) = \left(\int_M \frac{1}{\text{vol}(M)} |R|^p d\mu \right)^{1/p} = \left(\int_M \frac{1}{\text{vol}(M)} |R|^p u^{2^*} d\mu_0 \right)^{1/p},$$

and then try to pass to the limit as $p \rightarrow \infty$, using Lemma 2.3.2. That is, we first have to construct minimizers u_p for the minimization problem in L^p (which we will call p -problem). Then, we show that their limit exists and is the minimizer we want.

Initially, the existence of minimizers for the p -problem might seem an easy task, but an additional difficulty arises from the presence of the critical exponent 2^* , along with the constraints we plan to impose. This is caused by the lack of compactness in the Sobolev Embedding Theorem, as it was already demonstrated in Section 2.7. Moreover, since we are considering a variational problem in a geometric setting, there is always the possibility of the problem admitting a trivial solution. Firstly, observing that for any $\lambda \in \mathbb{R}$, we can consider the metric $g_\lambda = (\lambda u)^{2^*-2} g_0$, with curvature R^λ , we have:

$$R^\lambda = u^{1-2^*} \lambda^{1-2^*} (-c_n \Delta_{g_0}(u\lambda) + R_0(\lambda u)) =$$

$$u^{1-2^*} \lambda^{2-2^*} (-c_n \Delta_{g_0} u + Ru) = \lambda^{2-2^*} R,$$

after using the transformation law (2.4.4). Hence, because of this scaling property, the infimum we want to consider would be automatically zero, or not attained. In order to avoid this, we select a number $c_1 \in \mathbb{R}$, with $0 < c_1$, and require:

$$\mu(M) = c_1,$$

which is equivalent to asking:

$$\int_M d\mu = \int_M u^{2^*} d\mu_0 = c_1. \quad (3.1.1)$$

Additionally, there is always the possibility of the presence of a scalar-flat metric in our conformal class, that is a Riemannian metric $g_f \in [g_0]$ with $R_{g_f} = 0$. In particular, as we remarked in the Introduction and in Section 2.7, there are standard results of Escobar [13], extended by Marques in [24] and [25], guaranteeing the existence of a scalar-flat metric $g_f \in [g_0]$ for a compact Riemannian manifold with smooth boundary, in almost all cases. Moreover, this particular metric has constant mean curvature h_{g_f} . Independently of the latter property, our problem would a priori have a known solution with constant scalar curvature. Since we want to avoid a situation like that, we prescribe the average mean curvature of our conformal metric at the boundary, that is we select a number $c_2 \in \mathbb{R}$ such that:

$$\int_{\partial M} h_g d\sigma = \int_{\partial M} h_g u^{\frac{2(n-1)}{n-2}} d\sigma_0 = c_2. \quad (3.1.2)$$

Here σ is the surface measure corresponding to g , and σ_0 that corresponding to g_0 as usual. This constraint is helpful, since by (2.4.4) we have:

$$\int_M [c_n |\nabla^0 u|^2 + R_0 u^2] d\mu_0 - \int_{\partial M} c_n u \frac{\partial u}{\partial \nu_0} d\sigma_0 = \int_M R u^{2^*} d\mu_0 = \int_M R d\mu,$$

after multiplying our equation with u and integrating by parts.

Then we can use the transformation law (2.4.8) and constraint (3.1.2) to get:

$$\int_M [c_n |\nabla^0 u|^2 + R_0 u^2] d\mu_0 + 2(n-1) \left(\int_{\partial M} h_{g_0} u^2 d\sigma_0 - c_2 \right) = \int_M R d\mu.$$

So the factor c_2 here, can be used in order to get good bounds for u , in terms of the average mean curvature. Finally, we may remark that another result originating from Escobar again, might yield our problem trivial. Namely, in [14], it was proved that in most manifolds with boundary, there exists a constant scalar curvature metric, with zero mean curvature on the boundary. If that particular metric is scalar-flat, we still get a scalar-flat solution for our minimization problem. Nevertheless, we can avoid that situation, by choosing c_2 to be any nonzero real number.

In addition to the two integral constraints above, we also prescribe u along the boundary, with

$$u = u_0 \in C^\infty(\partial M). \quad (3.1.3)$$

This constraint might sound too restrictive, but normally one expects to have some stronger assumptions on the scalar and mean curvature, in order to solve a variational problem like that, see for example our discussion in Section 2.7. So, at the cost of only having an integral constraint on the mean curvature after a conformal change, we impose this stronger condition on the conformal factor over the boundary. Moreover, the Euler-Lagrange equations for our minimization problem turn out to be of fourth order, hence it is necessary to have two boundary conditions specified, so that we can get a well posed problem.

Note that we can alternatively use the following as a constraint over the boundary:

$$\int_{\partial M} u \frac{\partial u}{\partial \nu_0} d\sigma_0 = c_2,$$

instead of constraint (3.1.2), for $c_2 \in \mathbb{R}$. This particular condition might seem purely technical, but there is an extra geometric meaning, which becomes apparent by recalling the definition of the Yamabe energy $E_Y(u)$:

$$E_Y(u) = \int_M [c_n |\nabla^0 u|^2 + R_0 u^2] d\mu_0.$$

Then, after integrating by parts we can write:

$$E_Y(u) - \int_M R d\mu = c_n \int_{\partial M} u \frac{\partial u}{\partial \nu_0} d\sigma_0.$$

Hence, prescribing the term $\int_{\partial M} u \frac{\partial u}{\partial \nu_0} d\sigma_0$ can be thought of as prescribing the difference between the average scalar curvature and the Yamabe energy. We note that, in the case of manifolds without boundary $\int_M R d\mu = E_Y(u)$, since the boundary is empty, and there are no boundary integrals in the previous calculation. So, this type of constraint helps us make sure that the boundary of our manifold has an effect in our problem. Moreover, if we still keep the conformal factor u over the boundary ∂M prescribed, a constraint of this form, comes close to a constraint of the form:

$$\int_{\partial M} \frac{\partial u}{\partial \nu_0} d\sigma_0 = c,$$

for $c \in \mathbb{R}$, which is typical in that context.

Before proceeding to our main result, we briefly summarize our assumptions up to now, for the reader's convenience. We will attempt to minimize the functional:

$$E(u) = \|R\|_{L^\infty(M)},$$

using an approximation process and the Direct Method. The conformal factor u will be subject to the following constraints:

- $\int_M u^{2^*} d\mu_0 = c_1$, for $c_1 > 0$,
- $\int_{\partial M} h_g d\sigma_0 = c_2$, for $c_2 \in \mathbb{R}$,
- $u = u_0$ over ∂M , for $u_0 \in C^\infty(\partial M)$.

3.2 Statement of the main result

After having given the necessary constraints for our purposes, we can now proceed to specifying some subsets of Sobolev spaces to work with. For $\frac{n}{2} < p < \infty$, we define the subset $A^p(c_1, c_2, u_0)$ of the Sobolev space $W^{2,p}(M, g_0)$ by:

$$A^p(c_1, c_2, u_0) = \left\{ u \in W^{2,p}(M, g_0), \int_M u^{2^*} d\mu_0 = c_1, \right. \\ \left. \int_{\partial M} h_g d\sigma = c_2, u = u_0 \text{ on } \partial M, u > 0 \right\}.$$

Note that the normal derivative of u is well defined in $A^p(c_1, c_2, u_0)$. This follows from the fact that the trace of ∇u is well defined for $u \in W^{2,p}(M, g_0)$, under our assumptions on the regularity of ∂M (see [27] for example). Thus, the mean curvature of ∂M with respect to the metric g is well defined, since it is given by the formula:

$$\frac{n-2}{2}h_g u^{\frac{n}{n-2}} = \frac{n-2}{2}h_0 u + \frac{\partial u}{\partial \nu_0}.$$

Moreover, our condition on prescribing u over ∂M makes sense too, since u is continuous for $p > n/2$.

We also let $A^\infty(c_1, c_2, u_0)$, be the set of all $u \in \bigcap_{p < \infty} A^p(c_1, c_2, u_0)$, with curvature $R \in L^\infty(M, \mu_0)$. We will show that within A^∞ , there exists a minimizer for our problem, as long as it satisfies an upper bound on the energy $E(u)$.

In particular, with this notation in hand, our main result is as follows:

Theorem 3.2.1. *Let (M, g_0) a smooth compact Riemannian manifold with smooth boundary and dimension $n \geq 3$. Let $u_0 \in C^\infty(\partial M)$, $c_1 > 0$, $c_2 \in \mathbb{R}$, satisfy*

$$\inf_{u \in A^\infty(c_1, c_2, u_0)} E(u) < c_{crit}, \quad (3.2.1)$$

where

$$c_{crit} = c_{crit}(c_1, n) = \frac{c_n}{c_1^{2/n} K_n^2},$$

with K_n being the best constant for the Euclidean Sobolev inequality. Then, a minimizer u of E in $A^\infty(c_1, c_2, u_0)$ exists, with scalar curvature R satisfying $|R| = E(u)$, almost everywhere. Moreover, R is locally constant in $M \setminus \Gamma$, where Γ is a set contained in a countable union of embedded $n-1$ dimensional $C^{1,p}$ submanifolds and a closed $(n-2)$ -dimensional set.

This result is in line with the corresponding result in the case of surfaces in [26]. We remark that it is natural to impose some bounds on the infimum of $E(u)$, as we stressed in our discussion of the Yamabe problem. Moreover, c_{crit} appears elsewhere in the study of similar problems, as we will analyze in the sequel.

An interesting phenomenon is that the set Γ has another representation. It is the nodal set of the solution of a partial differential equation related to our

minimization problem. Remarking that we have:

$$E(u) = |R|,$$

almost everywhere for our minimizer u , we can see that a solution of our problem can be thought of as a solution of a Yamabe like problem. In particular, we recover a metric with constant scalar curvature, locally, up to sign, outside of a set with $\mu_0(\Gamma) = 0$. Moreover, we also have (3.1.2) still holding for the minimizer, thus we have it having constant average mean curvature over ∂M . Finally, we note that Γ is a countably $(n - 1)$ rectifiable set, following Lemma 2.6.2.

The proof of Theorem 3.2.1 takes up most of the remainder of this work, and is split into many individual parts. Hence, it is of benefit to give a brief summary of the strategy we will use again. We will first establish upper and lower bounds for solutions of equations like (2.4.4). Then, for fixed p bigger than a $q > 1$, existence of a minimizer u_p for the approximating p -problem is going to be established, by using the Direct Method. It is then natural to calculate the Euler-Lagrange equations for the p -problem, and then try to pass to the limit as $p \rightarrow \infty$. Following that, we study the limit Euler-Lagrange equation, in order to complete the proof of our Main Theorem.

CHAPTER 4

A PRIORI BOUNDS

In this Chapter, we will establish uniform upper and lower bounds for solutions of (2.4.4), under our constraints. Lower bounds are established first, taking advantage of our boundary condition (3.1.3). We then prove the existence of a uniform upper bound, by establishing suitable bounds on a family of curvature integrals. After that, we will define, and briefly discuss, the notion of concentration in volume and in curvature in our context.

4.1 Lower bounds

We will now establish lower bounds for solutions of (2.4.4), taking advantage of our boundary condition (3.1.3). We can see from (3.2.1), that there are some natural bounds on the L^p norms of the curvature functionals after a conformal change, for the class of functions that we study. We will take advantage of that, and the fact that we have our conformal factor prescribed over the boundary by (3.1.3). In particular, we can choose the minimum of our solution over the boundary to be as small as we like, and this is something we will exploit.

Proposition 4.1.1. *Let E_0 , $p > 0$ and $n \geq 3$, be such that $p > \frac{n}{2}$. Then, there exists a positive constant $C_2 = C_2(c_1, E_0, g_0, M, n, p, u_0)$, such that for every*

positive solution $u \in A^p(c_1, c_2, u_0)$ of the boundary value problem:

$$-c_n \Delta_{g_0} u + R_0 u = R u^{2^*-1}, \quad \text{in } M$$

$$u = u_0 \quad \text{on } \partial M,$$

with $E_p(u) < E_0$, it holds that:

$$u > C_2.$$

Proof. We note that since $u \in A^p(c_1, c_2, u_0)$ and $p > \frac{n}{2}$, we have $u \in C^0(\overline{M}, g_0)$ by the Sobolev Embedding Theorem 2.3.5. We can then define the following subset of M :

$$\Omega = \{x \in M \mid u(x) < c_0\},$$

for a fixed $c_0 > 0$, independent of u and chosen sufficiently small later on, such that:

$$\min_{\partial M} u_0 > c_0. \quad (4.1.1)$$

Note, that if $\Omega = \emptyset$, then $u \geq c_0$ in the whole of M . In that case $C_2 = c_0$, and there is nothing to prove. Moreover, Ω is an open subset of M as the preimage of $(0, c_0)$ under u .

In order to prove the existence of a lower bound in the case $\Omega \neq \emptyset$, we start by considering the following boundary value problem:

$$-c_n \Delta_{g_0} v + R_0^+ v = \begin{cases} R u^{2^*-1}, & \text{if } u < c_0 \text{ and } R < 0, \\ 0, & \text{else} \end{cases}$$

$$v = c_0 \text{ on } \partial M,$$

where R_0^+ stands for $R_0^+ = \max\{R_0, 0\}$ as usual.

Then, a solution $v \in W^{2,p}(M, g_0)$ to this problem always exists. This follows by the L^p theory for elliptic equations, as stated in Theorem 2.3.11, under our assumptions on the regularity of R_0 and R . In particular :

- $R_0^+ \in L^\infty(M, \mu_0)$, with $R_0^+ \geq 0$,
- $R u^{2^*-1}$ is an $L^p(M, \mu_0)$ function on the set $\{x \in M \mid u(x) < c_0, R(x) < 0\}$,

for $p > n/2$.

Furthermore, we can apply the maximum principle to our problem. In particular, we have:

$$c_n \Delta_{g_0} v - R_0^+ v \geq 0$$

in M , and $v = c_0$ on ∂M . Thus, the maximum principle, implies that:

$$v \leq c_0$$

in M , using the fact that $u > 0$.

Now we claim that $v \leq u$ in M . Indeed, in $M \setminus \Omega$ we have $c_0 \leq u$ by the definition of Ω , and since $v \leq c_0$, our claim follows. On the other hand, in Ω it holds that:

$$c_n \Delta_{g_0} v - R_0^+ v - c_n \Delta_{g_0} u + R_0^+ u = R_0^- u,$$

if $R < 0$, or

$$c_n \Delta_{g_0} v - R_0^+ v - c_n \Delta_{g_0} u + R_0^+ u = R_0^- u + R u^{2^*-1},$$

otherwise. So in any case:

$$c_n \Delta_{g_0} v - R_0^+ v - c_n \Delta_{g_0} u + R_0^+ u \geq 0$$

in Ω , since $u > 0$. Furthermore, on $\partial\Omega$ we have $v \leq u$ by our previous remark and the boundary conditions. Hence by the comparison principle, our claim follows.

We now consider the following boundary value problem:

$$c_n \Delta_{g_0} w - R_0^+ w = 0 \text{ in } M \tag{4.1.2}$$

$$w = c_0 \text{ on } \partial M.$$

A solution for (4.1.2) exists again, under our assumption by Theorem 2.3.11. We note that a simple application of the Strong Maximum Principle implies that our function cannot attain a non-positive minimum. Hence, there exists a $k > 0$ such that $w > k c_0$ in M . Here, k is independent of the boundary data, due to the

linearity of Poisson's equation (4.1.2).

Setting $\bar{v} = v - w$, we now have \bar{v} as a solution of the problem:

$$c_n \Delta_{g_0} \bar{v} - R_0^+ \bar{v} = \begin{cases} -Ru^{2^*-1} & \text{if } u < c_0 \text{ and } R < 0, \\ 0 & \text{else} \end{cases}$$

$$\bar{v} = 0 \text{ on } \partial M.$$

Then, the standard L^p regularity Theorem 2.3.11 implies that $\bar{v} \in W_0^{1,p}(M, g_0) \cap W^{2,p}(M, g_0)$.

It also holds that:

$$|\Delta_{g_0} \bar{v} - \frac{1}{c_n} R_0^+ \bar{v}| \leq \frac{1}{c_n} |R| c_0^{2^*-1}.$$

Raising to the power p ,

$$|\Delta_{g_0} \bar{v} - \frac{1}{c_n} R_0^+ \bar{v}|^p \leq (\frac{1}{c_n} |R| c_0^{2^*-1})^p,$$

and then integrating over M with respect to μ_0 , we have:

$$\int_M |\Delta_{g_0} \bar{v} - \frac{1}{c_n} R_0^+ \bar{v}|^p d\mu_0 \leq C c_0^{p(2^*-1)} \int_M |R|^p d\mu_0, \quad (4.1.3)$$

with $C = C(n, p)$. From the L^p theory for solutions of elliptic equations, and in particular from Theorem 2.3.11 again, we get an estimate for \bar{v} in $W^{2,p}(M, g_0)$. Our last estimate, combined with (4.1.3), yields the following inequality:

$$(\|\bar{v}\|_{W^{2,p}(M, g_0)})^p \leq C \int_M |\Delta_{g_0} \bar{v} - \frac{1}{c_n} R_0^+ \bar{v}|^p d\mu_0 \leq C c_0^{p(2^*-1)} \int_M |R|^p d\mu_0.$$

We then extend this inequality, for \bar{v} in $C^1(M, g_0)$, using the Sobolev Embedding Theorem 2.3.5. Thus, we conclude that :

$$(\sup |\bar{v}|)^p \leq C c_0^{p(2^*-1)} \int_M |R|^p d\mu_0,$$

where $C = C(M, n, p)$. This leads to the relation :

$$|v - w| \leq C \|R\|_{L^p(M, \mu_0)} c_0^{(2^*-1)},$$

after taking into account the definition of \bar{v} .

Let $\gamma = \inf_{x \in M} u$, with $\gamma > 0$, since u is positive and continuous on \bar{M} . Then, using the definition of E_p , and the fact that $u \in A^p(c_1, c_2, u_0)$, we have:

$$\begin{aligned} \|R\|_{L^p(M, \mu_0)} &\leq \left(\frac{1}{\gamma^{2^*}} \int_M |R|^p u^{2^*} d\mu_0 \right)^{1/p} = \gamma^{-2^*/p} c_1^{1/p} \left(\frac{1}{\text{vol}(M, \mu)} \int_M |R|^p u^{2^*} d\mu_0 \right)^{1/p} = \\ &\gamma^{-2^*/p} c_1^{1/p} E_p(u) \leq \gamma^{-2^*/p} c_1^{1/p} E_0. \end{aligned}$$

Hence, it follows that:

$$v > w - C \|R\|_{L^p(M, \mu_0)} c_0^{(2^*-1)} > c_0(k - C c_0^{2^*-2} c_1^{1/p} E_0 \gamma^{-2^*/p}),$$

which implies that:

$$u > c_0(k - C c_0^{2^*-2} c_1^{1/p} E_0 \gamma^{-2^*/p}),$$

since $u \geq v$. Thus we have:

$$\gamma > c_0(k - C c_0^{2^*-2} c_1^{1/p} E_0 \gamma^{-2^*/p}). \quad (4.1.4)$$

Setting $q = \frac{1}{2} + \frac{n}{4p}$, note that the following relation holds:

$$\frac{n}{2p} < q < 1,$$

since $p > n/2$. Moreover,

$$(2^* - 2)q - \frac{2^*}{p} = \frac{2 - \frac{n}{p}}{n - 2} > 0.$$

If $\gamma > (\inf_{\partial M} u_0)^{1/q}$, we have a uniform lower bound for u in \bar{M} , and there is nothing to prove. So, suppose that $\gamma \leq (\inf_{\partial M} u_0)^{1/q}$ and set $c_0 = \gamma^q$. Then, equation (4.1.4) implies that:

$$\gamma^{1-q} \geq (k - C c_1^{1/p} E_0 \gamma^{\frac{2-\frac{n}{p}}{n-2}}).$$

Clearly, a positive lower bound C_2 for u follows, as long as:

$$\gamma \geq \left(\frac{k}{2C c_1^{1/p} E_0} \right)^{\frac{n-2}{2-\frac{n}{p}}}.$$

Otherwise, the last inequality implies:

$$\gamma \geq \left(\frac{k}{2} \right)^{\frac{1}{1-q}},$$

allowing us to finish the proof in that case as well. \square

4.2 Upper bounds

In this section we will establish the existence of uniform upper bounds for solutions of our equation in \overline{M} . Following that, we will compare our results to existing results concerning the so called blow-up behavior of solutions of geometric equations. We begin with the statement of our Theorem.

Theorem 4.2.1. *Let $u \in A^{p_0}(c_1, c_2, u_0)$ be a positive solution of the equation:*

$$-c_n \Delta_{g_0} u + R_0 u = R u^{2^*-1}, \quad (4.2.1)$$

for $p_0 > n$ fixed. In addition, let δ , such that:

$$0 < \delta < \frac{c_n}{K_n^2}, \quad (4.2.2)$$

where K_n is the best constant for the Euclidean Sobolev Inequality. Then, if the following statement is true:

$$c_1^{\frac{2p_0}{n}-1} \int_M |R|^{p_0} u^{2^*} d\mu_0 \leq \delta^{p_0}, \quad (4.2.3)$$

there exists a positive constant $C = C(M, \delta, n, p_0, c_1, c_2, g_0, u_0)$, such that:

$$u(x) \leq C, \quad (4.2.4)$$

$\forall x \in \overline{M}$.

Proof. In order to reach a contradiction, suppose that condition (4.2.4) does not hold. Then, we can choose a sequence of solutions $\{u_\alpha\}_{\alpha \in \mathbb{N}}$ for (4.2.1), which satisfy (4.2.3), such that:

$$\sup_{x \in M} u_\alpha(x) \rightarrow \infty, \quad (4.2.5)$$

as $\alpha \rightarrow \infty$.

Nevertheless, note that we have some initial regularity results for solutions of (4.2.1). Indeed, multiplying equation (4.2.1) by u_α and integrating by parts yields:

$$c_n \int_M |\nabla u_\alpha|^2 d\mu_0 - c_n \int_{\partial M} u_\alpha \frac{\partial u_\alpha}{\partial \nu} d\sigma_0 + \int_M R_0 u_\alpha^2 d\mu_0 = \int_M R_\alpha u_\alpha^{2^*} d\mu_0.$$

Hence, we have using the triangle inequality :

$$\begin{aligned} & \left| c_n \int_M |\nabla u_\alpha|^2 d\mu_0 - c_n \int_{\partial M} u_\alpha \frac{\partial u_\alpha}{\partial \nu} d\sigma_0 \right| - \left| \int_M R_0 u_\alpha^2 d\mu_0 \right| \leq \\ & \left| c_n \int_M |\nabla u_\alpha|^2 d\mu_0 - c_n \int_{\partial M} u_\alpha \frac{\partial u_\alpha}{\partial \nu} d\sigma_0 + \int_M R_0 u_\alpha^2 d\mu_0 \right| = \left| \int_M R_\alpha u_\alpha^{2^*} d\mu_0 \right|, \end{aligned}$$

which immediately implies that:

$$\begin{aligned} & \left| c_n \int_M |\nabla u_\alpha|^2 d\mu_0 - c_n \int_{\partial M} u_\alpha \frac{\partial u_\alpha}{\partial \nu} d\sigma_0 \right| \leq \\ & \int_M |R_0| u_\alpha^2 d\mu_0 + \int_M |R_\alpha| u_\alpha^{2^*} d\mu_0. \end{aligned} \quad (4.2.6)$$

Note the following uniform bounds for the terms on the righthandside:

$$\int_M |R_0| u_\alpha^2 d\mu_0 \leq C \left(\int_M u_\alpha^{2^*} d\mu_0 \right)^{2/2^*} = C c_1^{2/2^*}, \quad (4.2.7)$$

$$\int_M |R_\alpha| u_\alpha^{2^*} d\mu_0 \leq \left(\int_M u_\alpha^{2^*} d\mu_0 \right)^{2/2^*} \left(\int_M |R_\alpha|^{n/2} u_\alpha^{2^*} d\mu_0 \right)^{2/n} \leq c_1^{2/2^*} \delta, \quad (4.2.8)$$

using our constraints, equation (4.2.3) and Holder's inequality with exponents $\frac{n}{2}$ and $\frac{2^*}{2}$. Here, and in what follows, C will denote a constant independent of α , unless otherwise stated. For the boundary term we have the following estimate,

by using the triangle inequality, Holder's inequality and our boundary conditions:

$$\begin{aligned}
\left| c_n \int_{\partial M} u_\alpha \frac{\partial u_\alpha}{\partial \nu} d\sigma_0 \right| &= \frac{n-2}{2} \left| c_n \int_{\partial M} (h_{g_\alpha} u_\alpha^{\frac{2(n-1)}{n-2}} - h_{g_0} u_\alpha^2) d\sigma_0 \right| = \\
&= 2(n-1) \left| \int_{\partial M} (h_{g_\alpha} u_0^{\frac{2(n-1)}{n-2}} - h_{g_0} u_0^2) d\sigma_0 \right| = \\
&= 2(n-1) \left| c_2 - \int_{\partial M} (h_{g_0} u_0^2) d\sigma_0 \right| \leq C.
\end{aligned} \tag{4.2.9}$$

Hence, from (4.2.6), (4.2.7), (4.2.8), (4.2.9) we have:

$$c_n \int_M |\nabla u_\alpha|^2 d\mu_0 \leq \int_M |R_0| u_\alpha^2 d\mu_0 + \int_M |R_\alpha| u_\alpha^{2^*} d\mu_0 + c \leq C.$$

So, we conclude that

$$\|u_\alpha\|_{W^{1,2}(M,g_0)} \leq C, \tag{4.2.10}$$

uniformly.

Let $x_\alpha \in \overline{M}$ such that $u_\alpha(x_\alpha) = \sup_{x \in M} u_\alpha(x)$. Also, let $\lambda_\alpha^{\frac{2-n}{2}} = u_\alpha(x_\alpha)$, and note that $\lambda_\alpha \rightarrow 0$ by our assumptions on u_α . Moreover, consider the following quantity:

$$d_\alpha = \frac{\text{dist}_{g_0}(x_\alpha, \partial M)}{\lambda_\alpha}.$$

Then we can assume that, up to choosing a subsequence, $d_\alpha \rightarrow d_0$, for some $d_0 \in [0, \infty]$. We will distinguish two cases in what follows, depending on the values of d_0 .

Case 1.

We first consider the case $d_0 \neq 0$, and initially assume, for the sake of presentation, that $d_0 = \infty$. Investigation of the case $0 < d_0 < \infty$ is postponed for the later part of our proof, since it involves some extra technical details. Note that by compactness, there exists a point $x_0 \in \overline{M}$ such that, up to selecting a subsequence, we have:

$$x_\alpha \rightarrow x_0,$$

as $\alpha \rightarrow \infty$. Our first step is to transfer equation (4.2.1) from M to some open

set in \mathbb{R}^n in a smooth way. In that direction, we consider a neighbourhood V of x_0 in \bar{M} , and a neighbourhood U of 0 in \mathbb{R}^n (if $x_0 \in M$), or in $\mathbb{R}^{n-1} \times [0, \infty)$ (if $x_0 \in \partial M$). Moreover, let Φ a smooth diffeomorphism:

$$\Phi : U \rightarrow V,$$

such that $\Phi(0) = x_0$. We also assume that:

$$(\Phi^* g_0)_{ij}(0) = \delta_{ij}, \quad (4.2.11)$$

where, as usual, we let $(g_0)_{ij}$ denote the components our metric. If $x_0 \in \partial M$, we require

$$\Phi(U \cap (\mathbb{R}^{n-1} \times \{0\})) = V \cap \partial M.$$

In addition, define \tilde{x}_α by:

$$\tilde{x}_\alpha = \Phi^{-1}(x_\alpha).$$

Then, for α large enough, we have $x_\alpha \in V$ and the sequence of functions

$$\hat{u}_\alpha(x) = \lambda_\alpha^{\frac{n-2}{2}} u_\alpha(\Phi(\lambda_\alpha x + \tilde{x}_\alpha))$$

is well defined in a ball $B_R(0)$ of radius $R > 0$ around 0, since $d_0 = \infty$.

The change of coordinates yields induced metrics $\hat{g}_\alpha(x) = \Phi^* g_0(\lambda_\alpha x + \tilde{x}_\alpha)$, with corresponding measures $\hat{\mu}_\alpha$, gradients $\hat{\nabla}^\alpha$ and Laplace-Beltrami operators $\Delta_{\hat{g}_\alpha}$. Then, in our new setting, the sequence \hat{u}_α satisfies the equation:

$$-c_n \Delta_{\hat{g}_\alpha} \hat{u}_\alpha + \hat{R}_{0\alpha} \hat{u}_\alpha = \hat{R}_\alpha \hat{u}_\alpha^{2^*-1}, \quad (4.2.12)$$

where

$$\begin{aligned} \bullet \quad \hat{R}_\alpha &= R_\alpha(\Phi(\lambda_\alpha x + \tilde{x}_\alpha)) \\ \bullet \quad \hat{R}_{0\alpha} &= \lambda_\alpha^2 R_0(\Phi(\lambda_\alpha x + \tilde{x}_\alpha)). \end{aligned} \quad (4.2.13)$$

This follows by the fact that u_α satisfies (4.2.1) and by simple scaling calculations.

In particular, at a point $x \in U$ we have:

$$\hat{u}_\alpha^{2^*-1}(x) = (\lambda_\alpha^{\frac{n-2}{2}} u_\alpha(\Phi(\lambda_\alpha x + \tilde{x}_\alpha)))^{2^*-1} = \lambda_\alpha^{\frac{n+2}{2}} (u_\alpha(\Phi(\lambda_\alpha x + \tilde{x}_\alpha)))^{2^*-1},$$

after using the definition of \hat{u}_α . Also, at a point $x \in U$ again, the Laplace-Beltrami operators corresponding to \hat{g}_α and g_0 transform according to:

$$\Delta_{\hat{g}_\alpha} \hat{u}_\alpha(x) = \lambda_\alpha^2 \Delta_{g_0} (\lambda_\alpha^{\frac{n-2}{2}} u_\alpha(\Phi(\lambda_\alpha x + \tilde{x}_\alpha))),$$

using the Chain Rule. Thus, the inclusion of the scaling factor λ_α^2 in the definition of $\hat{R}_{0\alpha}$ is justified. Moreover, we have $0 \leq \hat{u}_\alpha \leq 1$, with

$$\hat{u}_\alpha(0) = \lambda_\alpha^{\frac{n-2}{2}} u_\alpha(\Phi(\tilde{x}_\alpha)) = 1,$$

using the definition of Φ and that of \tilde{x}_α . Finally, the definition of our sequence of metrics implies that

$$\hat{g}_\alpha \rightarrow g_{euc}, \quad (4.2.14)$$

holds locally, where g_{euc} stands for the standard metric in \mathbb{R}^n .

The scaling construction that we are using, allows us to get some estimates on integral norms of \hat{u}_α over Euclidean balls. This happens as long as the exponents are chosen correctly. Specifically, it holds that:

$$\int_{B_R(0)} |\hat{\nabla}^\alpha \hat{u}_\alpha|^2 d\hat{\mu}_\alpha = \int_{\Phi(B_{\lambda_\alpha R}(\tilde{x}_\alpha))} |\nabla u_\alpha|^2 d\mu_0 \leq C, \quad (4.2.15)$$

and

$$\int_{B_R(0)} \hat{u}_\alpha^{2^*} d\hat{\mu}_\alpha \leq \int_{\Phi(B_{\lambda_\alpha R}(\tilde{x}_\alpha))} u_\alpha^{2^*} d\mu_0 \leq C, \quad (4.2.16)$$

for every radius $R > 0$ and for α large enough, after using the change of variables formula.

We now proceed by using a cut-off function argument as in [11], for a smooth function $0 \leq \eta \leq 1$, with:

$$\eta(x) = \begin{cases} 1 & \text{in } B_{R/2}(0), \\ 0 & \text{in } \mathbb{R}^n \setminus B_{3R/4}(0), \end{cases} \quad (4.2.17)$$

for some R independent of α . Then, letting $\eta_\alpha(x) = \eta(\lambda_\alpha x + \tilde{x}_\alpha)$, we note that by the definition of η we have:

$$|\hat{\nabla}^\alpha \eta_\alpha| \leq C \lambda_\alpha. \quad (4.2.18)$$

Following that, we derive some more estimates on $\eta_\alpha \hat{u}_\alpha$ in order to establish some regularity results. Trivially, we have:

$$\begin{aligned} \int_{\mathbb{R}^n} |\hat{\nabla}^\alpha(\eta_\alpha \hat{u}_\alpha)|^2 d\hat{\mu}_\alpha &= \int_{\mathbb{R}^n} |(\hat{\nabla}^\alpha \eta_\alpha) \hat{u}_\alpha|^2 d\hat{\mu}_\alpha + \int_{\mathbb{R}^n} |\eta_\alpha \hat{\nabla}^\alpha \hat{u}_\alpha|^2 d\hat{\mu}_\alpha + \\ 2 \int_{\mathbb{R}^n} \hat{\nabla}^\alpha \eta_\alpha \hat{u}_\alpha \hat{\nabla}^\alpha \hat{u}_\alpha \eta_\alpha d\hat{\mu}_\alpha &\leq 2 \left(\int_{\mathbb{R}^n} |\hat{\nabla}^\alpha \eta_\alpha|^2 |\hat{u}_\alpha|^2 d\hat{\mu}_\alpha + \int_{\mathbb{R}^n} |\hat{\nabla}^\alpha \hat{u}_\alpha|^2 |\eta_\alpha|^2 d\hat{\mu}_\alpha \right). \end{aligned} \quad (4.2.19)$$

Proceeding to a closer examination of the terms in (4.2.19), we obtain the following estimate:

$$\int_{\mathbb{R}^n} |\hat{\nabla}^\alpha \hat{u}_\alpha|^2 |\eta_\alpha|^2 d\hat{\mu}_\alpha \leq \int_{B_R(0)} |\hat{\nabla}^\alpha \hat{u}_\alpha|^2 d\hat{\mu}_\alpha \leq C,$$

for some $R > 0$, using (4.2.15). Moreover, if we turn our attention to the remaining term, we have:

$$\int_{\mathbb{R}^n} |\hat{\nabla}^\alpha \eta_\alpha|^2 |\hat{u}_\alpha|^2 d\hat{\mu}_\alpha \leq \left(\int_{\mathbb{R}^n} |\hat{\nabla}^\alpha \eta_\alpha|^n d\hat{\mu}_\alpha \right)^{2/n} \left(\int_{B_0(3/4\lambda_\alpha)} \hat{u}_\alpha^{2^*} d\hat{\mu}_\alpha \right)^{2/2^*} \leq C$$

using (4.2.16), (4.2.18) and Holder's inequality with exponents $\frac{n}{2}$ and $\frac{n}{n-2}$. Thus our two last estimates allow us to conclude that :

$$\int_{\mathbb{R}^n} |\hat{\nabla}^\alpha(\eta_\alpha \hat{u}_\alpha)|^2 d\hat{\mu}_\alpha \leq C.$$

This immediately implies that up to a subsequence:

$$\eta_\alpha \hat{u}_\alpha \rightharpoonup \hat{u}$$

for a limit function \hat{u} in $D_1^2(\mathbb{R}^n, \mu_{euc})$, since it is a reflexive space. Looking at (4.2.16) and taking into account the definition of η_α , we deduce that up to a subsequence:

$$\eta_\alpha \hat{u}_\alpha \rightharpoonup \hat{u}$$

in $L^{2^*}(\mathbb{R}^n, \mu_{euc})$ too. Also, note that for every $R > 0$:

$$\|\hat{u}\|_{L^{2^*}(B_R(0), \mu_{euc})} \leq \liminf_{\alpha \rightarrow +\infty} \|\eta_\alpha \hat{u}_\alpha\|_{L^{2^*}(B_R(0), \hat{\mu}_\alpha)} \leq C,$$

following the weak convergence, equations (4.2.14), (4.2.16) and the fact that $\eta \leq 1$. Here, the constant C is independent of R . Thus, we have:

$$\|\hat{u}\|_{L^{2^*}(\mathbb{R}^n, \mu_{euc})} \leq C.$$

We then claim that $\hat{u} \neq 0$. Indeed, writing:

$$-c_n \Delta_{\hat{g}_\alpha} \hat{u}_\alpha = (\hat{R}_\alpha \hat{u}_\alpha^{2^*-2} - \hat{R}_{0\alpha}) \hat{u}_\alpha$$

we will attempt to prove some good regularity properties for

$$f_\alpha = \hat{R}_\alpha \hat{u}_\alpha^{2^*-2} - \hat{R}_{0\alpha},$$

so that Lemma 2.3.12 can be used.

Initially, note that the inequalities:

$$\begin{aligned} \int_{B_R(0)} |\hat{R}_\alpha|^{n/2} \hat{u}_\alpha^{2^*} d\hat{\mu}_\alpha &= \int_{\Phi(B_{\lambda_\alpha R}(\tilde{x}_\alpha))} |R_\alpha|^{n/2} u_\alpha^{2^*} d\mu_0 \\ &\leq c_1^{1-\frac{n}{2p_0}} \left(\int_M |R_\alpha|^{p_0} u_\alpha^{2^*} d\mu_0 \right)^{\frac{n}{2p_0}} \leq \delta^{n/2}, \end{aligned} \quad (4.2.20)$$

is valid $\forall R > 0$, after using (4.2.3). Similarly, when $n/2 < s \leq p_0$ we have:

$$\int_{B_1(0)} |\hat{R}_\alpha \hat{u}_\alpha^{2^*-2}|^s d\hat{\mu}_\alpha \leq C,$$

after using Holder's inequality, along with the fact that $\hat{u}_\alpha \leq 1$. Moreover, for $n/2 < s \leq p_0$ again, we have:

$$\int_{B_1(0)} |\hat{R}_{0\alpha}|^s d\hat{\mu}_\alpha \leq C,$$

for a uniform constant C again. Thus, a bound:

$$\int_{B_1(0)} |f_\alpha|^s d\hat{\mu}_\alpha \leq C, \quad (4.2.21)$$

follows when $s \leq p_0$, and Lemma 2.3.12 is applicable. In particular, we derive:

$$1 = \sup_{B_{1/2}(0)} \hat{u}_\alpha \leq C \|\hat{u}_\alpha\|_{L^1(B_1(0), \hat{\mu}_\alpha)}.$$

It then follows that $\hat{u} \neq 0$, since

$$\hat{\eta}_\alpha \hat{u}_\alpha = \hat{u}_\alpha \rightarrow \hat{u}$$

strongly in $L^1(B_1(0))$, as $\alpha \rightarrow \infty$. Here, we used (4.2.17) in order to deduce that for α large enough $\hat{\eta}_\alpha \hat{u}_\alpha = \hat{u}_\alpha$ in $B_1(0)$.

Before the final part of our proof, we establish some convergence modes for \hat{R}_α too. Namely, up to a subsequence, it holds that:

$$\hat{R}_\alpha \hat{u}_\alpha^{2^*-2} \rightharpoonup f \quad \text{in} \quad L^{\frac{n}{2}}(\mathbb{R}^n, \mu_{euc}), \quad (4.2.22)$$

for a limit function f . This follows immediately after taking into account (4.2.3), (4.2.20), the fact that $\hat{u}_\alpha \leq 1$ and the reflexivity of $L^{n/2}$. Let $\phi \in C_0^\infty(\mathbb{R}^n)$, with $\phi \geq 0$ and $\text{supp} \phi \subset B_R(0)$. After multiplying (4.2.12) by $\eta_\alpha \phi$ and integrating by parts we have :

$$\begin{aligned} c_n \int_{B_R(0)} (\hat{\nabla}^\alpha \hat{u}_\alpha \hat{\nabla}^\alpha \phi) \eta_\alpha d\hat{\mu}_\alpha + c_n \int_{B_R(0)} (\hat{\nabla}^\alpha \hat{u}_\alpha \hat{\nabla}^\alpha \eta_\alpha \phi) d\hat{\mu}_\alpha \\ + \int_{B_R(0)} \hat{R}_{0\alpha} \hat{u}_\alpha \eta_\alpha \phi d\hat{\mu}_\alpha = \int_{B_R(0)} \hat{R}_\alpha \hat{u}_\alpha^{2^*-1} \eta_\alpha \phi d\hat{\mu}_\alpha. \end{aligned}$$

By the definition of $\hat{R}_{0\alpha}$, and since $\hat{u}_\alpha \leq 1$ we estimate:

$$\left| \int_{B_R(0)} \hat{R}_{0\alpha} \hat{u}_\alpha \eta_\alpha \phi d\hat{\mu}_\alpha \right| \leq C \max_M |R_0| \lambda_\alpha^2,$$

which goes to 0, as $\alpha \rightarrow +\infty$. Similarly, after using equation (4.2.18), we have

$$|\int_{B_R(0)} (\hat{\nabla}^\alpha \hat{u}_\alpha \hat{\nabla}^\alpha \eta_\alpha \phi) d\hat{\mu}_\alpha| \leq C\lambda_\alpha \int_{B_R(0)} |(\hat{\nabla}^\alpha \hat{u}_\alpha) \phi| d\hat{\mu}_\alpha,$$

with the latter expression tending to 0 as $\alpha \rightarrow \infty$. Thus for α large enough we have:

$$c_n \int_{B_R(0)} (\nabla^{Eu} \hat{u}_\alpha \nabla^{Eu} \phi) \eta_\alpha d\mu_{euc} = \int_{B_R(0)} \hat{R}_\alpha \hat{u}_\alpha^{2^*-1} \phi d\mu_{euc} + o(1).$$

Here ∇^{Eu} stands for the ordinary gradient in \mathbb{R}^n .

Using the weak convergence of $\hat{u}_\alpha \eta_\alpha$ in $D_1^2(\mathbb{R}^n)$, the corresponding strong result for L^p when $p < 2^*$, and taking advantage of the weak $L^{\frac{n}{2}}$ convergence that we have established for $\hat{R}_\alpha \hat{u}_\alpha^{2^*-2}$, we pass to the limit as $\alpha \rightarrow +\infty$ and after writing $R_\alpha u_\alpha^{2^*-1} = R_\alpha u_\alpha^{2^*-2} u_\alpha$ get:

$$c_n \int_{\mathbb{R}^n} (\nabla^{Eu} \hat{u} \nabla^{Eu} \phi) d\mu_{euc} \leq \int_{\mathbb{R}^n} f \hat{u} \phi d\mu_{euc}.$$

The denseness of $C_c^\infty(\mathbb{R}^n)$ in $D_1^2(\mathbb{R}^n, \mu_{euc})$ implies that we can find a sequence of functions $f_n \in C_c^\infty(\mathbb{R}^n)$ with $f_n \rightarrow \hat{u}$. So we can apply Holder's inequality with exponents $\frac{n}{2}$ and $2^*/2$, insert $\phi = \hat{u}$ and get:

$$c_n \int_{\mathbb{R}^n} |\nabla^{Eu} \hat{u}|^2 d\mu_{euc} \leq \int_{\mathbb{R}^n} f \hat{u}^2 d\mu_{euc} \leq \left(\int_{\mathbb{R}^n} |f|^{n/2} d\mu_{euc} \right)^{2/n} \left(\int_{\mathbb{R}^n} \hat{u}^{2^*} d\mu_{euc} \right)^{2/2^*}.$$

But from the definition of weak convergence and (4.2.20), we have:

$$\left(\int_{\mathbb{R}^n} |f|^{\frac{n}{2}} d\mu_{euc} \right)^{2/n} \leq \lim_{R \rightarrow \infty} \liminf_{\alpha \rightarrow +\infty} \|\hat{R}_\alpha \hat{u}_\alpha^{2^*-2}\|_{L^{\frac{n}{2}}(B_R(0), \hat{\mu}_\alpha)} \leq \delta.$$

Thus (4.2.2) immediately implies that:

$$\int_{\mathbb{R}^n} |\nabla^{Eu} \hat{u}|^2 d\mu_{euc} < \frac{1}{K_n^2} \left(\int_{\mathbb{R}^n} \hat{u}^{2^*} d\mu_{euc} \right)^{\frac{2}{2^*}}.$$

An argument based on [11], gives us a contradiction. In particular using the sharp Euclidean Sobolev Inequality Theorem 2.3.6:

$$\|\hat{u}\|_{L^{2^*}(\mathbb{R}^n)} \leq K_n \|\nabla^{Eu} \hat{u}\|_{L^2(\mathbb{R}^n)}$$

and our last inequality, we have:

$$\frac{(\int_{\mathbb{R}^n} \hat{u}^{2^*} d\mu_{euc})^{\frac{2}{2^*}}}{K_n^2} \leq \int_{\mathbb{R}^n} |\nabla^{Eu} \hat{u}|^2 d\mu_{euc} < \frac{(\int_{\mathbb{R}^n} \hat{u}^{2^*} d\mu_{euc})^{\frac{2}{2^*}}}{K_n^2},$$

which immediately yields a contradiction thus proving our argument, when $d_0 = \infty$.

In the case $0 < d_0 < \infty$, we only have to make some modifications in the preceding arguments. In particular, the definition of d_α , and the fact that d_0 is nonzero and finite, imply that up to a subsequence:

$$x_\alpha \rightarrow x_0 \in \partial M$$

holds, as α goes to ∞ . Thus, we need to consider boundary data in that case. For $x \in ((U \cap (\mathbb{R}^{n-1} \times \{0\}) - \tilde{x}_\alpha)/\lambda_\alpha)$, we have:

$$\hat{u}_\alpha(x) = \lambda_\alpha^{\frac{n-2}{2}} u_0(\Phi(\lambda_\alpha x + \tilde{x}_\alpha)),$$

using our prescribed function u_0 .

Employing the usual notation $x = (x', x_n)$, for $x \in \mathbb{R}^n$, we can write:

$$x_\alpha = (x'_\alpha, x_{n\alpha}).$$

Then, $\hat{u}_\alpha(x)$ is well defined when $x_n > -\frac{\tilde{x}_{n\alpha}}{\lambda_\alpha}$, and α is large enough. On the other hand, we can extend \hat{u}_α , by:

$$\lambda_\alpha^{\frac{n-2}{2}} u_0(\Phi(\lambda_\alpha x' + \tilde{x}'_\alpha, 0)), \text{ for } x_n \leq -\frac{\tilde{x}_{n\alpha}}{\lambda_\alpha}. \quad (4.2.23)$$

It follows that \hat{u}_α is well defined in $x \in B_R(0)$, for every $R > 0$, when α is sufficiently large. Moreover, the definition of \tilde{x}_α , and that of d_α , imply that, up to a subsequence:

$$\frac{\tilde{x}_{n\alpha}}{\lambda_\alpha} \rightarrow d_0, \quad (4.2.24)$$

as α goes to infinity.

Our previous arguments allow us now to deduce that:

$$-c_n \Delta_{\hat{g}_\alpha} \hat{u}_\alpha + \hat{R}_{0\alpha} \hat{u}_\alpha = \hat{R}_\alpha \hat{u}_\alpha^{2^*-1} \quad (4.2.25)$$

holds in the half-space $x_n > -\frac{\tilde{x}_{n\alpha}}{\lambda_\alpha}$. Here, the scaling relations (4.2.13) for $\hat{R}_{0\alpha}$ and \hat{R}_α are still valid in our new context. Also, inequalities (4.2.15) and (4.2.16) still hold, along with the definition of η_α in (4.2.17). Then, $\hat{u}_\alpha \eta_\alpha(x)$ is well defined in the whole \mathbb{R}^n as well, with:

$$\hat{u}_\alpha \eta_\alpha \rightharpoonup \hat{u},$$

for a limit function \hat{u} in $D_1^2(\mathbb{R}^n, \mu_{euc})$.

On the other hand, we have to modify some of our former considerations, in order to reach a contradiction again. In particular, we begin by letting:

$$f_\alpha(x) = \begin{cases} \hat{R}_\alpha \hat{u}_\alpha^{2^*-2} - \hat{R}_{0\alpha}, & \text{if } x_n > -\frac{\tilde{x}_{n\alpha}}{\lambda_\alpha}, \\ 0, & \text{if } x_n \leq -\frac{\tilde{x}_{n\alpha}}{\lambda_\alpha}. \end{cases}$$

Then, we only have to note that inequality (4.2.21) holds in $B_{d_0}(0)$. This implies that our Harnack inequality argument works in the case $0 < d_0 < \infty$ as well, when we work in a ball $B_r(0)$, of radius $r < d_0$. In particular, an estimate of the form:

$$1 \leq C \|\hat{u}_\alpha\|_{L^1(B_r(0), \hat{\mu}_\alpha)},$$

follows, when α is sufficiently large, and $r < d_0$. Thus, we are allowed to pass to the limit for $\eta_\alpha \hat{u}_\alpha$ in that case too.

Our final step consists of proving that (4.2.22) is valid in our new context. In that direction, we extend \hat{R}_α by zero, for $x_n \leq -\frac{\tilde{x}_{n\alpha}}{\lambda_\alpha}$. Then, it immediately follows that:

$$\hat{R}_\alpha \hat{u}_\alpha^{2^*-2} \rightharpoonup f \quad \text{in } L^{\frac{n}{2}}(\mathbb{R}^n, \mu_{euc}),$$

again. We can then choose test functions ϕ , with $\phi \in C_0^\infty(\mathbb{R}^{n-1} \times (-d_0, \infty))$. Multiplying (4.2.25) by $\eta_\alpha \phi$, as in the case $d_0 = \infty$, we are almost done. It then suffices to use (4.2.23) and (4.2.24), in order to conclude that $\hat{u} = 0$, when $x \in \mathbb{R}^{n-1} \times (-\infty, -d_0)$. Indeed, since $\lambda_\alpha \rightarrow 0$ and u_0 is independent of α , this

follows. The rest of our arguments from the first part of the proof, carry over under those modifications, thus yielding a contradiction too.

Case 2.

From now on, we work in the remaining case $d_0 = 0$. Note that the definition of d_α and the fact that $\lambda_\alpha \rightarrow 0$ guarantee that, up to extracting a subsequence,

$$x_\alpha \rightarrow x_0 \in \partial M.$$

The latter fact suggests that we should work in a domain with boundary in the n -dimensional Euclidean space, in order to reach a contradiction. Moreover, since ∂M is smooth, we can consider a boundary value problem in a domain $U \subset \overline{\mathbb{R}_+^n} = \mathbb{R}^{n-1} \times [0, \infty)$, if we apply a "straightening" process.

Similarly to the first part of this proof, we let Φ be a smooth diffeomorphism,

$$\Phi : U \rightarrow V,$$

where U is a set open relative to $\overline{\mathbb{R}_+^n}$ containing 0, and V an open subset of M , with $x_0 \in V$ and $\Phi(0) = x_0$. Moreover, we ask that Φ satisfies:

$$\Phi(U \cap (\mathbb{R}^{n-1} \times \{0\})) = V \cap \partial M.$$

and:

$$(\Phi^* g_0)_{ij}(0) = \delta_{ij}.$$

In that way, we locally straighten the boundary of U near $\Phi^{-1}(x_0)$, and make sure that our metric converges to the Euclidean one.

Let $\hat{u}_\alpha(x) = \lambda_\alpha^{\frac{n-2}{2}} u_\alpha(\Phi(\lambda_\alpha x))$. Note that \hat{u}_α is well-defined in a half ball $B_R^+(0)$ of radius $R > 0$ around 0, as long as α is large enough. If we define $\hat{R}_{0\alpha}$ and \hat{R}_α as in (4.2.13), but with \tilde{x}_α replaced by 0, then \hat{u}_α satisfies the following boundary value problem:

$$\begin{aligned} -c_n \Delta_{\hat{g}_\alpha} \hat{u}_\alpha + \hat{R}_{0\alpha} \hat{u}_\alpha &= \hat{R}_\alpha \hat{u}_\alpha^{2^*-1}, \text{ in } U/\lambda_\alpha, \\ \hat{u}_\alpha(x) &= \lambda_\alpha^{\frac{n-2}{2}} u_0(\Phi(\lambda_\alpha x)), \text{ in } (\partial U/\lambda_\alpha) \cap \{x_n = 0\}. \end{aligned} \quad (4.2.26)$$

Here x_n stands for the n -th coordinate in U . Then, letting $\hat{f}_\alpha = \hat{R}_\alpha \hat{u}_\alpha^{2^*-2} - \hat{R}_{0\alpha}$,

equation (4.2.26) can be written as:

$$\begin{aligned} -c_n \Delta_{\hat{g}_\alpha} \hat{u}_\alpha &= \hat{f}_\alpha \hat{u}_\alpha, \text{ in } U/\lambda_\alpha, \\ \hat{u}_\alpha(x) &= \lambda_\alpha^{\frac{n-2}{2}} u_0(\Phi(\lambda_\alpha x)), \text{ in } (\partial U/\lambda_\alpha) \cap \{x_n = 0\}. \end{aligned}$$

Note that for $x \in \partial U \cap \{x_n = 0\}$

$$\hat{u}_\alpha(x) \rightarrow 0, \tag{4.2.27}$$

as $\alpha \rightarrow \infty$. This follows since $\lambda_\alpha \rightarrow 0$ and our boundary data are independent of α . In addition, if we set:

$$\tilde{x}_\alpha = \lambda_\alpha^{-1} \Phi^{-1}(x_\alpha),$$

we still have $0 \leq \hat{u}_\alpha \leq 1$, with $\hat{u}_\alpha(\tilde{x}_\alpha) = 1$, as is evident from the definition of \hat{u}_α . Furthermore, the analogue of inequality (4.2.21) holds in our new context, that is we have:

$$\int_{B_1^+(0)} |\hat{f}_\alpha|^s d\hat{\mu}_\alpha \leq C,$$

for a fixed s , with $n/2 < s < p_0$. Note that the coefficients of $\Delta_{\hat{g}_\alpha}$ converge smoothly to those of the usual Euclidean Laplacian, as $\alpha \rightarrow 0$. This fact along with the smooth boundary data on $\partial U \cap \{x_n = 0\}$ allow us to use the standard elliptic regularity theory in $B_1^+(0)$. Hence, we first obtain a uniform bound for \hat{u}_α in $W^{2,p}(B_1^+(0))$, for $p > n/2$, after taking into account equation (4.2.10). The Sobolev Embedding Theorem then implies that:

$$\|\hat{u}_\alpha\|_{C^{0,\gamma}(B_1^+(0), \hat{g}_\alpha)} \leq C, \tag{4.2.28}$$

for some $0 < \gamma < 1$. Thus our sequence \hat{u}_α is equicontinuous. Taking into account that $\hat{u}_\alpha \leq 1$, allows us to deduce the existence of a function \tilde{u} :

$$u_\alpha \rightarrow \tilde{u},$$

uniformly as $\alpha \rightarrow \infty$, using the Arzela-Ascoli Theorem. Then, up to selecting a

subsequence, we have:

$$u_\alpha(\tilde{x}_\alpha) \rightarrow \tilde{u}(0) = 1,$$

uniformly as $\alpha \rightarrow \infty$. The latter clearly contradicts (4.2.27), thus concluding our proof in that case too. \square

4.3 Upper bounds and bubbling

Theorem 4.2.1 allows us to deduce bounds for solutions of equation (2.4.4), provided that we impose certain bounds on some curvature integrals. Similar results, under a bound on suitable L^p norms of scalar curvature functionals, have been derived elsewhere in the literature too (e.g [11], [29]). In most of the cases they are stated in the closed manifold setting. For the sake of completeness, we present one that is close to our situation. This happens so that we can discuss the differences with our approach.

We need to fix some notation beforehand, so let:

$$s = \int_M R d\mu$$

the total scalar curvature functional on M , and

$$F_p(g) = \int_M |R - s|^p d\mu.$$

Also let $s_* = \frac{c_n}{K_n^2}$, so that our notation is consistent with [29]. Moreover if for a sequence $\{u_k\}$, $\forall r > 0$ we have:

$$\liminf_{k \rightarrow \infty} \left(\int_{B_r(x)} |R_k|^{n/2} d\mu_k \right)^{2/n} \geq s_*, \quad (4.3.1)$$

at a point $x \in M$, we call x a concentration point. Similarly x is a concentration point for the volume functional, if for a sequence u_k , $\forall r > 0$ we have:

$$\liminf_{k \rightarrow \infty} \left(\int_{B_r(x)} u_k^{2^*} d\mu_k \right) \geq \alpha,$$

for some $\alpha > 0$.

The following Theorem is due to Schwetlick-Struwe [29] and describes the blow up behavior of a sequence of metrics under some bounds on the functionals s and F_p :

Theorem 4.3.1. *Let M a smooth, compact and closed manifold of dimension $n \geq 3$, with positive Yamabe invariant $\mathcal{Y}(M)$. Let $g_k = u_k^{2^*} g_0$ a sequence of metrics of unit volume, with $u_k \in C^\infty(M)$ positive and $C_0 > 0$ such that:*

$$F_p(g_k) \leq C_0 \quad s_k \leq C_0, \quad \forall k \in \mathbb{N}$$

for some $p > \frac{n}{2}$. Then either

1. u_k is uniformly bounded in $W^{2,p}(M, g_0) \hookrightarrow L^\infty(M, \mu_0)$,

or

2. $\exists(u_k)$ subsequence, $x_1, \dots, x_L \in M$, points in M , such that $\forall r > 0$, $l \in \{1, \dots, L\}$

$$\liminf_{l \rightarrow \infty} \left(\int_{B_r(x_l)} |R_k|^{n/2} d\mu_k \right)^{2/n} \geq s_*$$

and (u_k) is bounded in $W^{2,p}(K, g_0)$, on compact subsets $K \subset M \setminus \{x_1, \dots, x_L\}$.

As Schwetlick and Struwe point out in [29], if we work in balls B_r of radius r and volume less than 1, we have some relation between concentration in volume and concentration in curvature. In particular, using the triangle inequality, the Holder inequality, and the fact that $p > n/2$, we have

$$\begin{aligned} s_* &\leq \left(\int_{B_r(x_l)} |R_k|^{n/2} d\mu_k \right)^{2/n} = \left(\int_{B_r(x_l)} |R_k - s_k + s_k|^{n/2} d\mu_k \right)^{2/n} \\ &\leq \left(\int_{B_r(x_l)} |R_k - s_k|^{n/2} d\mu_k \right)^{2/n} + s_k \left(\int_{B_r(x_l)} d\mu_k \right)^{2/n} \\ &\leq \left(\int_{B_r(x_l)} |R_k - s_k|^p d\mu_k \right)^{1/p} \left(\int_{B_r(x_l)} d\mu_k \right)^{2/n-1/p} + s_k \left(\int_{B_r(x_l)} d\mu_k \right)^{2/n} \\ &\leq \left[\left(\int_{B_r(x_l)} |R_k - s_k|^p d\mu_k \right)^{1/p} + s_k \right] \left(\int_{B_r(x_l)} d\mu_k \right)^{2/n-1/p} \end{aligned}$$

$$\leq 2C_0 \left(\int_{B_r(x_l)} d\mu_k \right)^{2/n-1/p} \leq 2C_0 \left(\int_M d\mu_k \right)^{2/n-1/p}.$$

Thus, concentration in the sense of (4.3.1) implies concentration in volume.

In our case, we deduce a uniform lower bound for sequences of solutions of (2.4.4), allowing manifolds with boundary. Moreover, we do not impose any assumptions on some conformal invariant similar to $\mathcal{Y}(M)$.

CHAPTER 5

EXISTENCE FOR THE P-PROBLEM

This Chapter is devoted to the existence problem for the approximating p -problems for $p > p_0 > n$. At first, we prove that a minimizer exists using the Direct Method Theorem 2.7.1, and then calculate the Euler-Lagrange equation of the problem. Then we study the regularity of the solutions of this equation taking advantage of its form.

5.1 Existence of minimizers

Proposition 5.1.1. *Let $c_1 > 0$, $u_0 \in C^\infty(\overline{M})$ be given. Then $\forall \delta > 0$ and for every fixed p_0 with $p_0 > n$, such that:*

$$\delta < \frac{c_n}{K_n^2},$$

and

$$\inf_{u \in A^{p_0}(c_1, c_2, u_0)} E_{p_0}(u) < c_1^{1/p_0 - 2/n} \delta,$$

there exists a minimizer u_{p_0} of the functional $E_{p_0}(u)$ in the set $A^{p_0}(c_1, c_2, u_0)$.

Proof. We will use the Direct Method Theorem 2.7.1 to establish existence of a

minimizer for our minimization problem, for p_0 fixed. In that direction let

$$\{u_k\}_{k \in \mathbb{N}} \subset A^{p_0}(c_1, c_2, u_0)$$

a minimizing sequence for E_{p_0} , that is a sequence such that:

$$E_{p_0}(u_k) \rightarrow \inf_{u \in A^{p_0}(c_1, c_2, u_0)} E_{p_0}(u).$$

Initially we note that since

$$\int_M |R_k|^{p_0} u_k^{2^*} d\mu_0 < \delta^{p_0} c_1^{1-2p_0/n}, \quad (5.1.1)$$

we can use our a priori estimate analysis, for k sufficiently large. In particular Proposition 4.1.1 implies the existence of a lower bound C_2 for u_k and Theorem 4.2.1 that of an upper bound C_1 in \overline{M} . So when $p_0 > n$ there exist positive constants C_1 and C_2 independent of k , such that:

$$C_2 \leq u_k \leq C_1,$$

in \overline{M} . Moreover, we get the following integral bounds:

- $\|u_k\|_{L^{p_0}(M, \mu_0)} \leq C$, using our uniform upper bound for u_k ,
- $\|R_k u_k^{\frac{2^*}{p_0}}\|_{L^{p_0}(M, \mu_0)} \leq C$, since (5.1.1) holds,
- $\|u_0\|_{W^{2,p_0}(M, g_0)} \leq C$, using our boundary conditions,

for some constants C independent of k . Thus by Theorem 2.3.10 we have:

$$\|u_k\|_{W^{2,p_0}(M, g_0)} \leq C[\|u_k\|_{L^{p_0}(M, \mu_0)} + \|R_k u_k^{\frac{2^*}{p_0}}\|_{L^{p_0}(M, \mu_0)} + \|u_0\|_{W^{2,p_0}(M, g_0)}] \leq C,$$

with C independent of k . So up to a subsequence, it holds that:

$$u_k \rightharpoonup u$$

in $W^{2,p_0}(M, g_0)$, since it is a reflexive space, for a limit function u . Furthermore

we can use the Sobolev Embedding Theorem 2.3.5 to obtain a uniform bound

$$\|u_k\|_{1,\alpha(M,g_0)} \leq C,$$

for $\alpha \in (0, 1 - \frac{n}{p_0})$. So we have fulfilled part of the assumptions of Theorem 2.7.1 and it remains to examine the convergence modes that we get from the boundedness properties we established.

We can assume that, up to a subsequence again, $u_k \rightarrow u$ in $C^1(\overline{M}, g_0)$, using the compactness of the Sobolev Embedding $W^{2,p}(M, g_0) \hookrightarrow C^1(M, g_0)$. Hence, we conclude that $u \neq 0$, after taking into account Proposition 4.1.1 too. But, we also have $R_k \rightharpoonup \hat{R}$ weakly in $L^q(M, \mu_0)$, up to a subsequence, for a limit curvature \hat{R} and every $q < \infty$. So, we need to show that $\hat{R} = R$ is indeed the curvature of the minimizer u to our problem.

Taking into account the definition of weak convergence in L^q spaces, we have $R_k \rightharpoonup \hat{R}$ if

$$\int_M R_k \phi d\mu_0 \rightarrow \int_M \hat{R} \phi d\mu_0,$$

for every $\phi \in L^{q'}(M, \mu_0)$, where q' is the conjugate exponent of q . It follows from the definition of convergence in $C^1(\overline{M}, g_0)$ that we have:

- $u_k \rightarrow u$
- $\partial^j u_k \rightarrow \partial^j u, \forall j < \infty$

uniformly on compact subsets of \overline{M} . Moreover we can use the denseness of smooth compactly supported functions in $L^{q'}(M, \mu_0)$. Then by taking into account the definition of R_k we have:

$$\begin{aligned} \int_M R_k u_k^{2^*-1} \phi d\mu_0 &= \int_M [-c_n \Delta_{g_0} u_k + R_0 u_k] \phi d\mu_0 = \\ &= -c_n \int_M u_k \Delta_{g_0} \phi d\mu_0 + \int_M R_0 u_k \phi d\mu_0, \end{aligned}$$

after integrating by parts and taking advantage of the fact that $\phi \in C_0^\infty(M, g_0)$. In addition we know that:

$$|R_0 u_k| \leq C$$

and

$$|u_k \Delta_{g_0} \phi| \leq C$$

with constants independent of k by Theorem 4.2.1. Hence, we can use the strong C^1 convergence of our sequence on compact subsets and the Dominated Convergence Theorem, to pass to the limit and deduce that:

$$\begin{aligned} -c_n \int_M \phi \Delta_{g_0} u d\mu_0 + \int_M R_0 u \phi d\mu_0 &= -c_n \int_M u \Delta_{g_0} \phi d\mu_0 + \int_M R_0 u \phi d\mu_0 \\ &= \lim_{k \rightarrow \infty} [-c_n \int_M u_k \Delta_{g_0} \phi d\mu_0 + \int_M R_0 u_k \phi d\mu_0] = \lim_{k \rightarrow \infty} \int_M R_k u_k^{2^*-1} \phi d\mu_0 \\ &= \int_M \lim_{k \rightarrow \infty} (R_k u_k^{2^*-1} \phi) d\mu_0 = \int_M \hat{R} u^{2^*-1} \phi d\mu_0, \end{aligned}$$

proving that $\hat{R} = R$.

The lower semicontinuity of the L^{p_0} norm in M with respect to μ_0 , allows to conclude that

$$\begin{aligned} E_{p_0}(u) &= \left(\int_M \frac{1}{c_1} |R|^{p_0} u^{2^*} d\mu_0 \right)^{1/p_0} \leq \liminf_{k \rightarrow \infty} \left(\int_M \frac{1}{c_1} |R_k|^{p_0} u_k^{2^*} d\mu_0 \right)^{1/p_0} \\ &= \liminf_{k \rightarrow \infty} E_{p_0}(u_k). \end{aligned}$$

So, we then deduce that the infimum of E_{p_0} is achieved, using the lower semicontinuity of the functional. Nevertheless, we need to perform a number of additional steps before concluding our proof. We have to verify that $u \in A^{p_0}(c_1, c_2, u_0)$ too, that is our constraints are preserved when passing to the limit as $k \rightarrow \infty$.

Firstly, Theorem 4.2.1, and the C^1 -convergence we established, imply that the volume constraint is preserved, since we have:

$$u_k \rightarrow u$$

strongly in $L^{2^*}(M, \mu_0)$. In addition, our boundary condition of prescribing $u = u_0$ on ∂M , implies that the minimizer satisfies this constraint on the boundary too. Hence, we only have to verify that (3.1.2) is preserved. If h_k is the mean curvature

of the metric g_k , then our boundary condition for u_k can be written in terms of:

$$\frac{n-2}{2}h_k u_0^{\frac{2(n-1)}{n-2}} = \frac{\partial u_k}{\partial \nu_0} u_0 + \frac{n-2}{2}h_0 u_0^2,$$

by the fact that u_k is prescribed over the boundary. It is then sufficient to examine closely the term $\frac{\partial u_k}{\partial \nu_0} u_0$, and from the strong C^1 convergence that we have established in \overline{M} it follows that:

$$h_k \rightarrow h = u_0^{-\frac{2(n-1)}{n-2}} \left(\frac{\partial u}{\partial \nu_0} u_0 + \frac{n-2}{2}h_0 u_0^2 \right)$$

on ∂M , uniformly as $k \rightarrow \infty$, thus concluding our proof. \square

5.2 The Euler-Lagrange equations

In this section, we will explicitly calculate the Euler-Lagrange equations for our minimization problem. Since we have already established existence of minimizers for $p > n$, we make our calculations for $p \in (n, \infty)$ fixed.

We want to minimize the functional $E_p(u)$, in the class $\mathcal{A}^p(c_1, c_2, u_0)$, that is subject to the constraints:

1. $V(u) := \int_M d\mu = \int_M u^{2^*} d\mu_0 = c_1$
2. $J(u) := \int_{\partial M} h_g d\sigma = c_2,$
3. $u = u_0 \in C^\infty(\partial M)$ over ∂M ,

where $c_1, c_2 \in \mathbb{R}$ with $c_1 > 0$.

Naturally, we will compute the first variation of each of those functionals for $\phi \in W_0^{1,p}(M) \cap W^{2,p}(M, g_0)$ an admissible variation, with ϕ vanishing at the boundary, due to our constraint of u being prescribed over the boundary. Firstly, after taking into account the transformation formula (2.4.4), and denoting by R_t the curvature belonging to $u + t\phi$, we have:

$$R_t = (u + t\phi)^{1-2^*} L_{g_0}(u + t\phi), \tag{5.2.1}$$

where L_{g_0} is the conformal Laplacian of g_0 . Consequently, it holds that:

$$\begin{aligned} \frac{d}{dt}|_{t=0} \int_M |R_t|^p d\mu &= \frac{d}{dt}|_{t=0} \int_M |R_t|^p (u + t\phi)^{2^*} d\mu_0 = \\ &= \int_M p |R|^{p-2} R [(1 - 2^*)(L_{g_0} u)\phi + u L_{g_0} \phi] + 2^* |R|^p u^{2^*-1} \phi d\mu_0, \end{aligned}$$

where $\phi \in W_0^{1,p}(M) \cap W^{2,p}(M, g_0)$. For the functional V we have the following:

$$\frac{d}{dt}|_{t=0} V(u + t\phi) = \frac{d}{dt}|_{t=0} \left[\int_M (u + t\phi)^{2^*} d\mu_0 \right] = 2^* \int_M u^{2^*-1} \phi d\mu_0, \quad (5.2.2)$$

for $\phi \in W_0^{1,p}(M) \cap W^{2,p}(M, g_0)$. Finally, for the average mean curvature functional J , we have:

$$\begin{aligned} J(u) &= \int_{\partial M} h_g d\sigma = \int_{\partial M} h_g u^{\frac{2(n-1)}{n-2}} d\sigma_0 = \\ &= \int_{\partial M} h_0 u^2 d\sigma_0 + \frac{2}{n-2} \int_{\partial M} u \frac{\partial u}{\partial \nu_0} d\sigma_0, \end{aligned}$$

after taking into account the transformation formula (2.4.8) and the fact that $d\sigma = u^{\frac{2(n-1)}{n-2}} d\sigma_0$. Hence, the first variation of J is given by:

$$\frac{d}{dt}|_{t=0} (J(u + t\phi)) = 2 \int_{\partial M} h_0 u \phi d\sigma_0 + \frac{2}{n-2} \int_{\partial M} \frac{\partial u}{\partial \nu_0} \phi d\sigma_0 + \frac{2}{n-2} \int_{\partial M} \frac{\partial \phi}{\partial \nu_0} u d\sigma_0, \quad (5.2.3)$$

for all $\phi \in W_0^{1,p}(M) \cap W^{2,p}(M, g_0)$.

It is now time to use the Lagrange multipliers rule, in order to deduce the actual Euler Lagrange equations for our problem. For the minimization of $E_p(u)$ under our constraints, there exist Lagrange multipliers $a, b \in \mathbb{R}$, such that if u_p is a minimizer satisfying our constraints, and R_p is the scalar curvature of the metric

$$g_p = u_p^{2^*-2} g_0,$$

we have:

$$\int_M p |R_p|^{p-2} R_p [(1 - 2^*) L_{g_0} u_p \phi + u_p L_{g_0} \phi] + 2^* |R_p|^p u_p^{2^*-1} \phi d\mu_0 - a 2^* \int_M u_p^{2^*-1} \phi d\mu_0$$

$$-b[2 \int_{\partial M} h_0 u_p \phi d\sigma_0 + \frac{2}{n-2} \int_{\partial M} \frac{\partial u_p}{\partial \nu_0} \phi d\sigma_0 + \frac{2}{n-2} \int_{\partial M} \frac{\partial \phi}{\partial \nu_0} u_p d\sigma_0] = 0,$$

for all $\phi \in W_0^{1,p}(M) \cap W^{2,p}(M, g_0)$. This follows after using equations (5.2.1), (5.2.2) and (5.2.3).

We will now continue with some formal calculations, so that we transform the integral identities related to our Euler Lagrange equations into a nicer form. We begin by writing

$$L_{g_0} \phi = -c_n \Delta_{g_0} \phi + R_0 \phi,$$

and getting:

$$\begin{aligned} & \int_M p |R_p|^{p-2} R_p [(1-2^*) L_{g_0} u_p \phi + u_p R_0 + 2^* |R_p|^p u_p^{2^*-1} - a 2^* u_p^{2^*-1}] \phi d\mu_0 \\ & - \int_M [p c_n |R_p|^{p-2} R_p u_p] \Delta_{g_0} \phi d\mu_0 - b[2 \int_{\partial M} h_0 u_p \phi d\sigma_0 \\ & + \frac{2}{n-2} \int_{\partial M} \frac{\partial u_p}{\partial \nu_0} \phi d\sigma_0 + \frac{2}{n-2} \int_{\partial M} \frac{\partial \phi}{\partial \nu_0} u_p d\sigma_0] = 0. \end{aligned}$$

Using Green's Formula twice, we get:

$$\begin{aligned} & - \int_M [p c_n |R_p|^{p-2} R_p u_p] \Delta_{g_0} \phi d\mu_0 = - \int_{\partial M} \frac{\partial \phi}{\partial \nu_0} [c_n p |R_p|^{p-2} R_p u_p] d\sigma_0 \\ & + \int_M \nabla [p c_n |R_p|^{p-2} R_p u_p] \nabla \phi d\mu_0 = - \int_{\partial M} \frac{\partial \phi}{\partial \nu_0} [c_n p |R_p|^{p-2} R_p u_p] d\sigma_0 \\ & - \int_M \Delta_{g_0} [p c_n |R_p|^{p-2} R_p u_p] \phi d\mu_0 + \int_{\partial M} \frac{\partial [p c_n |R_p|^{p-2} R_p u_p]}{\partial \nu_0} \phi d\sigma_0. \end{aligned}$$

Hence, our Euler Lagrange equation, can now be written as:

$$\begin{aligned} & \int_M (p |R_p|^{p-2} R_p [(1-2^*) L_{g_0} u_p + u_p R_0] + 2^* |R_p|^p u_p^{2^*-1}) \phi d\mu_0 \\ & - \int_M [a 2^* u_p^{2^*-1} + \Delta_{g_0} (p c_n |R_p|^{p-2} R_p u_p)] \phi d\mu_0 \\ & - \int_{\partial M} \frac{\partial \phi}{\partial \nu_0} [p c_n |R_p|^{p-2} R_p u_p] d\sigma_0 \end{aligned}$$

$$\begin{aligned}
& + \int_{\partial M} \frac{\partial [pc_n |R_p|^{p-2} R_p u_p]}{\partial \nu_0} \phi d\sigma_0 - b[2 \int_{\partial M} h_0 u_p \phi d\sigma_0 \\
& + \frac{2}{n-2} \int_{\partial M} \frac{\partial u_p}{\partial \nu_0} \phi d\sigma_0 + \frac{2}{n-2} \int_{\partial M} \frac{\partial \phi}{\partial \nu_0} u_p d\sigma_0] = 0.
\end{aligned}$$

Since we have prescribed our function at the boundary, we get that $\phi = 0$ necessarily on ∂M , so that the weak form of the Euler - Lagrange equation can be simplified more, and be written as:

$$\begin{aligned}
& \int_M (p|R_p|^{p-2} R_p [(1-2^*)L_{g_0} u_p + u_p R_0] + 2^*|R_p|^p u_p^{2^*-1}) \phi d\mu_0 \\
& - \int_M [a2^* u_p^{2^*-1} + \Delta_{g_0} (pc_n |R_p|^{p-2} R_p u_p)] \phi d\mu_0 \\
& - \int_{\partial M} \frac{\partial \phi}{\partial \nu_0} [pc_n |R_p|^{p-2} R_p u_p] d\sigma_0 - \frac{2b}{n-2} \int_{\partial M} \frac{\partial \phi}{\partial \nu_0} u_p d\sigma_0 = 0. \tag{5.2.4}
\end{aligned}$$

Choosing a ϕ with vanishing normal derivative at the boundary of M , we have:

$$\begin{aligned}
& p|R_p|^{p-2} R_p [(1-2^*)L_{g_0} u_p + u_p R_0] + 2^*|R_p|^p u_p^{2^*-1} \\
& - a2^* u_p^{2^*-1} - \Delta_{g_0} (pc_n |R_p|^{p-2} R_p u_p) = 0
\end{aligned}$$

holding in M . Now, using the definition of L_{g_0} , and that

$$L_{g_0} u_p = R_p u_p^{2^*-1},$$

by the transformation law (2.4.4), we get:

$$p(1-2^*)|R_p|^p u_p^{2^*-1} + 2^*|R_p|^p u_p^{2^*-1} + pL_{g_0}(|R_p|^{p-2} R_p u_p) = a2^* u_p^{2^*-1}.$$

Multiplying by $u_p^{1-2^*}$, we have:

$$p(1-2^*)|R_p|^p + 2^*|R_p|^p + p u_p^{1-2^*} L_{g_0}(|R_p|^{p-2} R_p u_p) = a2^*. \tag{5.2.5}$$

We will use the transformation law (2.5.2), in order to further simplify our equation. If

$$L_{g_p} = -c_n \Delta_{g_p} + R_p$$

is the conformal Laplacian of the metric $g_p = u_p^{2^*} g_0$ and Δ_{g_p} the respective Laplace-Beltrami operator, then from (2.5.2) it follows that:

$$u_p^{1-2^*} L_{g_0}(|R_p|^{p-2} R_p u_p) = L_{g_p}(|R_p|^{p-2} R_p) = -c_n \Delta_g(|R_p|^{p-2} R_p) + |R_p|^p.$$

Consequently, we have :

$$p((2 - 2^*) + 2^* |R_p|^p) - p c_n \Delta_g(|R_p|^{p-2} R_p) = a 2^*,$$

and dividing by $c_n p$, we get:

$$\left[\frac{n}{2p(n-1)} - \frac{1}{n-1} \right] |R_p|^p - \Delta_g(|R_p|^{p-2} R_p) = a 2^* / p c_n := a_p.$$

Then, using the fact that equation (5.2.4) holds for all $\phi \in W_0^{1,p}(M, g_0) \cap W^{2,p}(M, g_0)$, we get :

$$p c_n (|R_p|^{p-2} R_p) = -\frac{2}{n-2} b, \quad \text{on } \partial M.$$

Setting $b_p := -\frac{2}{(n-2)p c_n} b$, we have:

$$\begin{aligned} -\Delta_{g_p}(|R_p|^{p-2} R_p) + \left[\frac{n}{2p(n-1)} - \frac{1}{n-1} \right] |R_p|^p &= a_p, \quad \text{on } M \\ |R_p|^{p-2} R_p &= b_p, \quad \text{on } \partial M. \end{aligned}$$

We remark, that the condition $p > n$, guarantees that $\frac{n}{2p(n-1)} - \frac{1}{n-1} \neq 0$. Our problem is of fourth order in u_p , but we will express it as a second order one, after some convenient renormalization. In that direction we can define:

$$\gamma_p = \max\{|a_p|, |b_p|, \|R_p\|_{L^p(\mu_p)}^{p-1}\}, \quad (5.2.6)$$

with $\alpha_p = a_p / \gamma_p$, $\beta_p = b_p / \gamma_p$ and $w_p = \frac{|R_p|^{p-2} R_p}{\gamma_p}$. We assume that $\gamma_p \neq 0$ here, as the case γ_p is trivial. It follows that our initial boundary value problem can now be reformulated in the weak sense as:

$$-\Delta_{g_p} w_p + \left[\frac{n}{2p(n-1)} - \frac{1}{n-1} \right] R_p w_p = \alpha_p \quad \text{on } M$$

$$w_p = \beta_p \quad \text{on } \partial M, \quad (5.2.7)$$

concluding our calculations.

5.3 Uniform Estimates

In this section, we assume that the hypotheses of Theorem 3.2.1 are satisfied. We will establish some regularity results for minimizers u_p of E_p , under the constraints (3.1.1), (3.1.2) and (3.1.3). In addition, we let R_p be the scalar curvature of the metric $g_p = u_p^{2^*-2}g_0$ and μ_p the corresponding measure on M . We remark once more that our boundary value problem:

$$\begin{aligned} -\Delta_{g_p} w_p + \left[\frac{n}{2p(n-1)} - \frac{1}{n-1} \right] R_p w_p &= \alpha_p \quad \text{on } M \\ w_p &= \beta_p \quad \text{on } \partial M, \end{aligned}$$

is of fourth order in u_p , but its form allows us to treat it like a second order one. We also stress the definition of w_p as :

$$w_p = \frac{|R_p|^{p-2} R_p}{\gamma_p}, \quad (5.3.1)$$

and the fact that R_p is defined through a second order equation itself. Moreover, our construction by normalization, guarantees that $\alpha_p, \beta_p \in [-1, 1]$. Hence, it suffices to examine the other terms of our equations closer, in order to establish some regularity results for w_p .

In that direction, we remark that since the hypotheses of Theorem 3.2.1 are satisfied, there exists a minimizer u_p for E_p , satisfying equation (3.2.1). Thus, we have:

$$\limsup_{p \rightarrow \infty} \|R_p\|_{L^p(M, \mu_p)} < c_{crit},$$

with $c_{crit} = \frac{c_n}{c_1^{2/n} K_n^2}$. Thus, equation (4.2.3) is satisfied for some $\delta < \frac{c_n}{K_n^2}$, when p is sufficiently large, and we can apply Theorem 4.2.1 and Proposition 4.1.1. So, we can deduce that there exist constants $C_1, C_2 > 0$, which are independent of p ,

such that:

$$C_1 \leq u_p \leq C_2,$$

for p large enough. In particular, it holds that the norms of $L^q(M, \mu_0)$ and $L^q(M, \mu_p)$ are equivalent for any $1 \leq q \leq \infty$. Hence, we can conclude that $R_p \rightharpoonup \hat{R}$ in $L^q(M, \mu_0)$, for a sequence $p \rightarrow \infty$ and a function \hat{R} , since R_p is a bounded sequence in a reflexive space.

Also, from the definition of w_p , we have :

$$\begin{aligned} \|w_p\|_{L^{p'}(M, \mu_p)} &= \left(\int_M |w_p|^{p'} d\mu_p \right)^{1/p'} = \left(\int_M \left| \frac{|R_p|^{p-2} R_p}{\gamma_p} \right|^{p'} d\mu_p \right)^{1/p'} = \\ &= \left(\int_M \frac{[|R_p|^{p-2} R_p]^{p'}}{(\gamma_p)^{p'}} d\mu_p \right)^{1/p'} \leq \frac{1}{\gamma_p} \|R_p\|_{L^p(M, \mu_p)}^{p-1} \leq 1, \end{aligned}$$

by our renormalization (5.2.6). Using Theorem 2.3.10 and the definition of R_p , we deduce that the sequence u_p is bounded in $W^{2,q}(M, g_0)$, for any $q < \infty$. Then, since $W^{2,q}(M, g_0) \hookrightarrow C^1(\overline{M})$ compactly from the Sobolev Embedding Theorem 2.3.5, we can assume without loss of generality, that up to a subsequence:

$$u_p \rightharpoonup u \text{ in } W^{2,q}(M, g_0) \text{ and } u_p \rightarrow u \text{ in } C^1(\overline{M}).$$

Since R_p is uniformly bounded in $L^p(M, \mu_p)$ and w_p is uniformly bounded in $L^{p'}(M, \mu_p)$, we can conclude using Holder's inequality that $R_p w_p \in L^1(M, \mu_p)$ is uniformly bounded. Here μ_p corresponds to the measure arising from g_p . Defining:

$$k_p := \alpha_p - \left[\frac{n}{2p(n-1)} - \frac{1}{n-1} \right] R_p w_p,$$

where $k_p =: k(n, p)$, only depends on n and p , we now have to study a boundary value problem of the form:

$$-\Delta_p w_p = k_p \quad \text{on } M$$

$$w_p = \beta_p \quad \text{on } \partial M,$$

with the right hand side of the equation being uniformly bounded in $L^1(M, \mu_0)$.

The standard elliptic regularity theory Theorem 2.3.11 is of no use at the moment, since it could only be applied in the case we had $R_p w_p \in L^{q_0}(M, \mu_0)$ for some $q_0 > 1$. If that was true, then we would get some uniform bounds for w_p in $W^{2,q_0}(M, g_0)$, for some $q_0 > 1$, and then a standard bootstrapping argument could be applied. Nevertheless, it is possible to use an alternative scheme to get some better initial regularity for w_p , based on [15].

We begin, by interpreting this boundary value problem in the weak sense again, thus having:

$$-\int_M w_p \Delta_p \phi d\mu_p = \int_M k_p \phi d\mu_p,$$

where $\phi \in C_c^\infty(M)$. Now, taking advantage of the fact that: $k_p \in L^1(M, \mu_0)$, we can interpret $k_p \mu_0$ as a bounded sequence of Radon measures μ_{k_p} . A standard compactness result in that case ([15]) leads us to the conclusion that $w_p \in W^{1,q}(M, g_0)$, for every $1 \leq q < \frac{n}{n-1}$, with the inequality being strict. Moreover, our uniform L^1 bound for $R_p w_p$ implies that w_p is uniformly bounded in $W^{1,q}(M, g_0)$ too.

We can now proceed to the derivation of some better regularity properties of w_p , by proving the following:

Lemma 5.3.1. *Let w_p a sequence of solutions of the boundary value problem:*

$$-\Delta_g w_p + \left[\frac{n}{2p(n-1)} - \frac{1}{n-1} \right] R_p w_p = \alpha_p \quad \text{on } M$$

$$w_p = \beta_p \quad \text{on } \partial M,$$

with $w_p \in W^{1,q}(M, g_0)$, uniformly bounded $\forall q$, with $1 < q < \frac{n}{n-1}$. Then, w_p is uniformly bounded in $W^{2,\tilde{q}}(M, g_0)$, for all $\tilde{q} < \frac{n}{n-2}$.

Proof. We begin by using the Sobolev Embedding Theorem 2.3.5, in order to conclude that w_p is uniformly bounded in $L^{\tilde{p}}(M, \mu_0)$, for every $\tilde{p} = \frac{nq}{n-q}$, with $q < \frac{n}{n-1}$. Thus, it follows that:

$$w_p \in L^{\tilde{p}}(M, \mu_0),$$

and is uniformly bounded, for every $\tilde{p} < \frac{n}{n-2}$.

Now, we claim that $R_p w_p \in L^\lambda(M, \mu_0)$, for every $\lambda < n/(n-2)$ and for p large enough. This follows from a direct application of Holder's inequality, since $R_p \in L^p(M, \mu_0)$. Moreover, we deduce bounds that are independent p . Furthermore, we can use the standard L^p theory of elliptic equations of second order Theorem 2.3.10, since we have the leading order coefficients of Δ_{g_p} uniformly bounded in $C^{1,\alpha}(M, g_0)$, for some $\alpha \in (0, 1)$. Thus, we conclude that $w_p \in W^{2,q}(M, g_0)$, for every $q < \frac{n}{n-2}$ and

$$\|w_p\|_{2,q} \leq c(\|R_p w_p\|_\lambda + \beta_p) \leq C,$$

with C independent of p . Hence, w_p is uniformly bounded in $W^{2,q}(M, g_0)$, for every $q < \frac{n}{n-2}$. \square

In order to obtain further regularity results, bootstrapping is needed, hence we have the following:

Lemma 5.3.2. *Let $l \in \mathbb{N}$, with $l \geq 2$. Then the sequence w_p , with $w_p = \frac{|R_p|^{p-2} R_p}{\gamma_p}$, is uniformly bounded in $L^q(M, \mu_0)$, for every $q < \frac{n}{n-l}$, provided that $l < n$.*

Proof. We will prove our claim using induction in l .

In the base case $l = 2$, our last Lemma 5.3.1 and the Sobolev Embedding Theorem 2.3.5, guarantee that w_p is uniformly bounded in $L^q(M, \mu_0)$, for every $q < \frac{n}{n-2}$.

Now, suppose that for $l_0 \in \mathbb{N}$, we have w_p uniformly bounded in $L^q(M, \mu_0)$ with

$$q < \frac{n}{n-l_0}.$$

For the induction step, suppose that:

$$l_0 + 1 < n,$$

implying that $l_0 < n - 1$. Now by our inductive hypothesis, we get a uniform bound for w_p in $L^q(M, \mu_0)$, for every $q < \frac{n}{n-l_0}$. Then, a uniform bound for $R_p w_p$ in $L^{\tilde{q}}(M, \mu_0)$ follows by using Holder's inequality, when $\tilde{q} < \frac{n}{n-l_0}$. Moreover, the

partial differential equation for w_p , combined with the standard elliptic regularity Theorem 2.3.10, provide us with a uniform bound for w_p in $W^{2,\tilde{q}}(M, g_0)$, when $\tilde{q} < \frac{n}{n-l_0}$ again. Then, the Sobolev Embedding Theorem 2.3.5 implies a uniform bound for every

$$q < \frac{n}{n-l_0-2} = \frac{n}{n-(l_0+2)} > \frac{n}{n-(l_0+1)},$$

proving our claim. □

Lemma 5.3.3. *The sequence w_p defined by $w_p = \frac{|R_p|^{p-2}R_p}{\gamma_p}$ is uniformly bounded in $C^{0,\alpha}(M, g_0)$, $\forall \alpha < 1$ and converges up to a subsequence, uniformly to a limit function w .*

Proof. From the Sobolev Embedding Theorem 2.3.5, it suffices to prove that w_p is uniformly bounded in $W^{2,\tilde{q}}(M, g_0)$ for $\frac{n}{2} < \tilde{q} < n$. But by our previous Lemma, we can always achieve uniform $L^{\tilde{q}}$ bounds, for $\frac{n}{2} < \tilde{q}$, by letting $l \in \mathbb{N}$, be sufficiently large, if necessary. A standard application of Theorem 2.3.10 provides us with uniform bounds in $W^{2,\tilde{q}}(M, g_0)$, thus proving our claim.

Now, w_{p_k} is a uniformly bounded sequence in $C^{0,\alpha}(M, g_0)$, which is equicontinuous. Thus, the Arzela-Ascoli Theorem applies, and there exists a subsequence of w_p , which we also call w_p , which converges uniformly to a limit w , concluding our proof. □

CHAPTER 6

PROOF OF THE MAIN RESULT

This Chapter consists of the completion of the proof of Theorem 3.2.1. It is separated into two smaller sections, the first one being the study of the limit Euler-Lagrange equation for our problems, and a second one related to the nodal set of its solutions.

6.1 The Limit Equation

We begin this section by summarizing our results up to now. We proved in Chapter 5, that there exists a positive minimizer u_p of E_p , for every $p \in (n, \infty)$. Moreover, the metric $g_p = u_p^{2^*-2}g_0$ has scalar curvature R_p , given by:

$$R_p = u_p^{1-2^*}[-c_n\Delta_{g_0}u_p + R_0u_p].$$

Then, we proved that the solutions w_p of the corresponding Euler-Lagrange equations of our p -minimization problem satisfy certain regularity results. In this section, we will use the regularity properties that we previously established for w_p , in order to derive a limit equation and obtain a minimizer for our problem, thus proving Theorem 3.2.1. So, up to a subsequence $p_k \rightarrow \infty$, we know that

1. $u_{p_k} \rightharpoonup u$ in $W^{2,q}(M, g_0)$ and $u_{p_k} \rightarrow u$ in $C^1(\overline{M}, g_0)$,

2. $w_{p_k} \rightarrow w$ uniformly from Lemma 5.3.3,
3. $R_{p_k} \rightharpoonup \hat{R}$ in $L^q(M, \mu_0)$ for every $q < \infty$.

Then:

$$R_{p_k} = u_{p_k}^{1-2^*} [-c_n \Delta_{g_0} u_{p_k} + R_0 u_{p_k}] \rightharpoonup \hat{R},$$

so we can deduce that:

$$\hat{R} = R = u^{1-2^*} [-c_n \Delta_{g_0} u + R_0 u]. \quad (6.1.1)$$

Since $w_{p_k} \rightarrow w$ uniformly for the same sequence p_k , using the boundary value problem (5.2.7), after letting $p \rightarrow \infty$, we have:

$$\begin{aligned} -\Delta_g w - \frac{1}{n-1} R w &= \alpha, \quad \text{in } M \\ w &= \beta, \quad \text{on } \partial M, \end{aligned} \quad (6.1.2)$$

in the weak sense. Here α, β are the limits of $\alpha_{p_k}, \beta_{p_k}$ respectively and Δ_g is the Laplacian of the metric $g = u^{2^*-2} g_0$. We will first prove that we are in a nontrivial situation, that is w is not identically zero in \overline{M} , as proven in the next:

Lemma 6.1.1. *The limit w of the subsequence w_{p_k} , defined by $w_{p_k} = \frac{|R_{p_k}|^{p_k-2} R_{p_k}}{\gamma_p}$, is not identically equal to zero under our assumptions.*

Proof. We already know that w satisfies:

$$\begin{aligned} -\Delta_g w - \frac{1}{n-1} R w &= \alpha, \quad \text{in } M \\ w &= \beta, \quad \text{on } \partial M. \end{aligned}$$

We will show that it is not identically equal to zero, provided that $(\alpha, \beta) \neq (0, 0)$, or $\|w\|_{L^1(M, \mu)} \neq 0$. Indeed, depending on the value of the maximum γ_{p_k} , we have either $(\alpha, \beta) \neq (0, 0)$, or $\|w\|_{L^1(M, \mu)} = 1$. In particular recalling that :

$$\gamma_{p_k} = \max \{ |a_{p_k}|, |b_{p_k}|, \|R_{p_k}\|_{L^{p_k}(M, \mu_{p_k})}^{p_k-1} \},$$

if our maximum is either $|a_{p_k}|$ or $|b_{p_k}|$, then either $|\alpha_{p_k}|$ or $|\beta_{p_k}| = 1$, for k large enough. So in that case, $\alpha \neq 0$ implies that w is not 0 identically zero in the interior, and $\beta \neq 0$ that $w \neq 0$ on the boundary. Finally, if $\gamma_{p_k} = \|R_{p_k}\|_{L^{p_k}(M, \mu_{p_k})}^{p_k-1}$, then by definition:

$$(w_{p_k})^{\frac{p_k}{p_k-1}} = \frac{|R_{p_k}|^{p_k}}{\|R_{p_k}\|_{L^{p_k}(M, \mu_{p_k})}^{p_k}},$$

hence $\|w_{p_k}\|_{L^{p'_k}(M, \mu_{p_k})} = 1$, for k large enough. The uniform convergence of w_{p_k} allows us to deduce that $\|w\|_{L^1(M, \mu)} = 1$, proving the fact that $w \neq 0$ in that case too, thus concluding our proof. \square

We recall once more that:

$$|w_{p_k}|^{\frac{1}{p_k-1}} = \frac{|R_{p_k}|}{\gamma_{p_k}^{\frac{1}{p_k-1}}}, \quad (6.1.3)$$

from the definition of w_{p_k} . Then, defining $\Gamma = w^{-1}(\{0\})$ and letting $k \rightarrow \infty$, we have:

$$\log |w_{p_k}|^{\frac{1}{p_k-1}} \rightarrow 0,$$

outside of Γ , since $w_{p_k} \rightarrow w$ uniformly. This in turns implies:

$$|w_{p_k}|^{\frac{1}{p_k-1}} \rightarrow 1,$$

as $k \rightarrow \infty$, locally uniformly in $M \setminus \Gamma$. In addition, up to a subsequence, we have:

$$\gamma_{p_k}^{1/(p_k-1)} \rightarrow \gamma_\infty,$$

as $k \rightarrow \infty$, for some $\gamma_\infty \in [0, \infty]$. Hence, from (6.1.3), it holds that $|R| = \gamma_\infty$ in $M \setminus \Gamma$. Note that the set Γ is closed relative to M , as the intersection of the closed set Γ with M . In addition, $R < \infty$ obviously, from equation (6.1.1). Then, since $w \neq 0$ and $\Gamma \neq M$, we are able to deduce $\gamma_\infty < \infty$.

We recall the definition of the functional E_p :

$$E_p(u) = \left(\frac{1}{c_1} \int_M |R|^p d\mu\right)^{1/p},$$

where c_1 is the volume of M , with respect to μ_p . From the definition of u_p :

$$E_p(u_p) \leq E_p(u_q),$$

for $p \leq q$, and by Holder's inequality:

$$\begin{aligned} E_p(u_q) &= \left(\frac{1}{c_1} \int_M |R|^p d\mu_q \right)^{1/p} \leq \left(\frac{1}{c_1} \right)^{1/p} c_1^{\frac{q-p}{qp}} \left(\int_M |R|^q d\mu_q \right)^{1/q} \leq \\ &\quad \left(\frac{1}{c_1} \right)^{1/q} \left(\int_M |R|^q d\mu_q \right)^{1/q} = E_q(u_q). \end{aligned}$$

Hence, $\lim_{p \rightarrow \infty} E_p(u_p) = e_\infty$ exists, and from the lower semicontinuity of E_p , and the definition of \liminf

$$E_q(u) \leq \liminf_{k \rightarrow \infty} E_q(u_{p_k}) \leq \liminf_{k \rightarrow \infty} E_{p_k}(u_{p_k}) = e_\infty. \quad (6.1.4)$$

We also remark that u belongs to $A^\infty(c_1, c_2, u_0)$ too. Indeed, the volume constraint (3.1.1) is satisfied for u_{p_k} , and is preserved as we pass to the limit. In particular, following the strong C^1 convergence that we have established in \overline{M} , it holds that :

$$u_{p_k} \rightarrow u$$

strongly in L^{2^*} . Moreover, the boundary constraint (3.1.3) is preserved too, since we have prescribed boundary values for our sequence. In addition, the mean curvature of the metric $u_{p_k}^{2^*-2} g_0$ is given by:

$$\frac{n-2}{2} h_{p_k} u_0^{\frac{2(n-1)}{n-2}} = \frac{\partial u_{p_k}}{\partial \nu_0} u_0 + \frac{n-2}{2} h_0 u_0^2.$$

The strong C^1 convergence that we have established in \overline{M} guarantees that:

$$h_{p_k} \rightarrow h = u_0^{-\frac{2(n-1)}{n-2}} \left(\frac{2}{n-2} \frac{\partial u}{\partial \nu_0} u_0 + h_0 u_0^2 \right),$$

uniformly on ∂M , implying that (3.1.2) is preserved too.

Since the L^∞ norm can be obtained by the L^p norm, following Lemma 2.3.2, letting $p \rightarrow \infty$, we conclude that $E_\infty(u) \leq e_\infty$. On the other hand, since u_p

minimizes E_p , we have:

$$e_\infty \leq \lim_{p \rightarrow \infty} E_p(u_p) \leq \lim_{p \rightarrow \infty} E_p(\tilde{u}) = E_\infty(\tilde{u}), \quad (6.1.5)$$

for any other $\tilde{u} \in A^\infty(c_1, c_2, u_0)$. Now from (6.1.4) and (6.1.5), we can conclude that u is a minimizer for our problem in $A^\infty(c_1, c_2, u_0)$.

6.2 The Nodal Set of the Solution

In this section, we will study the nodal set $\Gamma = w^{-1}\{0\}$ of the solution to the limit boundary value problem:

$$\begin{aligned} -\Delta_g w - \frac{1}{n-1} R w &= \alpha, & \text{in } M \\ w &= \beta, & \text{on } \partial M. \end{aligned}$$

More precisely, we will prove that the set Γ has the structure stated in Theorem 3.2.1, using a result from [19]. There, the authors prove that we have good control on the size of the nodal set of a solution to an elliptic equation in a bounded domain under certain assumptions. In particular, as long as we have good control on the coefficients of the equation and on the zero set of the gradient of the solution, we get better regularity for the nodal set. Moreover, if we write $\Gamma = N(w) \cup S(w)$, with $N(w) = \{x \in \Gamma, Dw \neq 0\}$ and $S(w) = \Gamma \setminus N(w)$, then we have regularity results for both $N(w)$ and $S(w)$. This will allow us to complete the proof of our main Theorem.

We begin by establishing some more regularity results for w . From the Sobolev Embedding Theorem 2.3.5, we know that u is uniformly bounded in $C^{1,\alpha}(M, g_0)$, $\forall \alpha \in (0, 1)$, so that by Definition 2.3.8 of the Laplace-Beltrami operator, its leading order coefficients belong to $C^{1,\alpha}(M)$, $\forall \alpha \in (0, 1)$. Moreover, the term $\frac{1}{n-1} R w$ belongs to L^∞ . Hence, we can deduce that $w \in C^{1,\alpha}(M, g_0)$, $\forall \alpha \in (0, 1)$.

In addition, from the previous section, we know that $Rw = \gamma_\infty |w|$, so taking the term $\frac{1}{n-1} R w$ to the right handside in equation (6.1.2), we can conclude using the Schauder Theory Theorem 2.3.9 that $w \in C^{2,\alpha}(M, g_0)$, $\forall \alpha \in (0, 1)$, because

the right handside is Lipschitz continuous. We have the following, even better, result on the regularity of w :

Proposition 6.2.1. *The function w is locally smooth, on each one of the sets M^+ , M^- , where $M^+ = \{x \in M, w(x) > 0\}$ and $M^- = \{x \in M, w(x) < 0\}$.*

Proof. The set M^+ is open, as the preimage of $(0, +\infty)$ under w . Writing $w = w^+ - w^-$, our boundary value problem now implies that:

$$-\Delta_g w^+ - \frac{1}{n-1} R w^+ = \alpha, \quad \text{on } M^+, \quad (6.2.1)$$

and

$$\Delta_g w^- + \frac{1}{n-1} R w^- = \alpha, \quad \text{on } M^-.$$

In M^+ , we have $w = w^+$, hence by the relation $Rw = \gamma_\infty |w|$, which holds by the definition of w in the whole M , we can deduce that $R = \gamma_\infty$. Consequently:

$$-\Delta_g w^+ = \alpha + \frac{1}{n-1} \gamma_\infty w^+, \quad (6.2.2)$$

holds on M^+ . We can then take advantage of the fact that $w^+ \in C_{loc}^{2,\rho}(M^+)$, for all $\rho \in (0, 1)$. The right handside of (6.2.1) belongs to $C_{loc}^{2,\rho}(M)$ too and we can use Schauder Theory to deduce that $w^+ \in C_{loc}^{4,\rho}(M^+)$, for all $\rho \in (0, 1)$. Iterating this estimate, we obtain the desired result for w^+ , with the case of w^- being completely analogous. \square

We now state and prove a result on the form and regularity of the nodal set of Γ , which will allow us to complete the proof of Theorem 3.2.1.

Proposition 6.2.2. *The set $\Gamma = w^{-1}\{0\}$ is contained in the union of a countable union of embedded $C^{1,\rho}$ submanifolds and a countably $(n-2)$ rectifiable closed set.*

Proof. If $\alpha = 0$, we have w satisfying the following equation

$$-\Delta_g w - \frac{1}{n-1} R w = 0,$$

on M . We write $\Gamma = N(w) \cup S(w)$, where $N(w) = \{x \in \Gamma, Dw \neq 0\}$ and $S(w) = \Gamma \setminus N(w)$. Then 0 is a regular value of w for $x \in N(w)$, hence we can use

the Implicit Function Theorem, to deduce that $N(w)$ is contained in the union of countably many $C^{2,\rho}$ manifolds of dimension $n - 1$.

For $S(w)$, we use the fact that $w \in C^{2,\rho}(M)$, and then a result of Hardt and Simon [19] applies. Namely, we have $S(w)$ contained in a countable union of subsets of a pairwise disjoint collection of smooth $(n - 2)$ dimensional submanifolds. Then, using Lemma 2.6.2, we conclude that $S(w)$ is countably $(n - 2)$ rectifiable.

Similarly, if $\alpha \neq 0$ the method used for $N(w)$ in the first part of the proof still applies, hence we only have to prove the corresponding result for $S(w)$. We have w satisfying

$$-\Delta_g w - \frac{1}{n-1} R w = \alpha,$$

on M . Thus, $-\Delta_g w(x) = \alpha$ for $x \in S(w)$, and this implies that $d\frac{\partial w}{\partial x_i}(x) \neq 0$, for some $i \leq n$, where $\{x_i\}$ are local coordinates centered around a point in $S(w)$. Consequently, $S(w)$ is contained in the union of countably many $(n-1)$ dimensional $C^{1,\rho}$ manifolds by the Implicit Function Theorem, since each $x \in S(w)$ is a regular value for $\frac{\partial w}{\partial x_i}$. \square

Hence, in any case we can conclude that $\mu_0(\Gamma) = 0$, due to the structure of Γ , as presented in the last lemma. Then, it follows that $E(u) = \gamma_\infty$. Also, recall that $|R| = \gamma_\infty$ in $M \setminus \Gamma$. Thus, we have:

$$|R| = E(u),$$

in $M \setminus \Gamma$, finishing our proof.

6.3 Outlook

The nature of our geometric problem allows us to deduce boundedness of a solution to equation (2.4.4) under some particular constraints, and a bound on our energy $E(u)$. Nevertheless, it would be interesting to study our problem under different constraints in order to see differences in the results we obtain. In particular one could try to replace constraint (3.1.3) by prescribing the normal derivative of u over the boundary. In that case we can still obtain similar Euler-

Lagrange equations similar to those in Chapter 5, but our approach to the existence of a lower bound will not be valid. Moreover it is not known whether we can prove similar results to our Main Theorem 3.2.1 without the presence of the upper bound on the energy $E(u)$, a concern which is close to the ones expressed in [26]. Finally it would be interesting to try to establish existence of minimizers to our problem using parabolic methods. There are still some difficulties though stemming from the fact that our equations are of fourth order.

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